SYNTHESIS OF FOUR-BAR LINKAGES BY ITERATION OF THE CLOSURE EQUATIONS

A THESIS

Presented to

the Faculty of the Graduate Division

by

Herbert Hewett Hill

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

June, 1961

4 8
12.T

SYNTHESIS OF FOUR-BAR LINKAGES BY ITERATION OF THE CLOSURE EQUATIONS

APPROVED:

Date Approved by Chairman: August 1, 1961

ACKNOWLEDGMENTS

I wish to express my sincere thanks to Dr. Joseph P. Vidosic, Regents' Professor of Mechanical Engineering, my thesis advisor; to Dr. Homer S. Weber, Director of the School of Mechanical Engineering; to Dr. B. M. Drucker, Associate Professor of the School of Mathematics; and to Dr. Karl M. Murphy, Associate Professor of the English Department, for making possible this research and for aiding me in carrying it out.

TABLE OF CONTENTS

TABLE OF CONTENTS (Concluded)

LIST OF ILLUSTRATIONS

LIST OF TABLES

GLOSSARY

$\mathcal{Z}_1$	Ground link of 4-bar mechanism
$\mathcal{Z}_2$	Crank of 4-bar mechanism
$\mathcal{Z}_3$	Coupler of 4-bar mechanism
$\mathcal{Z}_4$	Follower of 4-bar mechanism
α_{12}	Crank rotation from position 1 to position 2
α_{13}	Crank rotation from position 1 to position 3
α_{14}	Crank rotation from position 1 to position 4
α_{15}	Crank rotation from position 1 to position 5
β_{12}	Coupler rotation from position 1 to position 2
β_{13}	Coupler rotation from position 1 to position 3
β_{14}	Coupler rotation from position 1 to position 4
β_{15}	Coupler rotation from position 1 to position 5
σ_{12}	Follower rotation from position 1 to position 2
σ_{13}	Follower rotation from position 1 to position 3
σ_{14}	Follower rotation from position 1 to position 4
σ_{15}	Follower rotation from position 1 to position 5
$\mathcal{R}_\emptyset$	$\cos \emptyset + i \sin \emptyset$
$\lambda_\emptyset$	$(\cos \emptyset - 1) + i \sin \emptyset$
N_a	Real (horizontal) component of link a: a = 1,2,3, or 4
$i\eta_a$	Imaginary (vertical) component of link a: a = 1,2,3, or 4

SUMMARY

The accelerated interest in function generation since World War II has resulted in the development of a variety of methods for this type of synthesis. For the purpose of providing another method of synthesizing 4-bar linkages as function generators, a study was conducted of the simultaneous solution of closure equations by iterative means.

A 4-bar linkage was represented by components involving complex numbers, from which the closure equations were derived. It was observed when these equations were written as sets, representing the initial and subsequent positions, that there must exist some inter-relation between the rotational variables. To show the inter-relation for a mechanism in four positions, an analysis was made of three closure equations written in determinant form. Then different combinations of the rotational ranges were chosen as parameters and, by iteration, mechanisms were found whose links satisfied the given closure equations. Nine trials were conducted and the results analyzed graphically to determine which combinations were satisfactory for this iterative method. In the ninth trial a most effective combination was found between the range of the coupler and the ranges of the other links.

A method for finding a linkage satisfying five positions was developed as a continuation of the four positions case by a different derivation of the closure equations, which included the coupler as a variable. An analytical method was also derived for determining the follower rotation when the crank is rotated its assumed full range. This analytical method enabled a computer program to be written which will find a mechanism satisfying five positions.

As an example of this method, two different 4-bar linkages were synthesized satisfying a given function for five positions.

Finally, a step-by-step description is presented which should enable a designer familiar with function generation to use the method very easily.

CHAPTER I

BRIEF HISTORY OF FOUR-BAR LINKAGES

It is not exactly clear when the first 4-bar linkage was developed, but it was well over 2000 years ago. The original notes of Archimedes (1)* describe inventions, such as catapults and cranes, which he built for King Hiero, all utilizing levers and types of 4-bar linkages.

Ever since that time, man has been interested in ways of transferring and converting motion, and 4-bar linkages have adapted readily to many of these types of mechanization. For instance, Grodzinski and McEwen (2) stated at the Institute of Mechanical Engineers' Meeting in 1954, "Link mechanisms, consisting of substantially rigid members connected by simple turning or sliding pairs, are not only the simplest and oldest type of mechanisms, but are fundamental to any more complex machines." However, the synthesis of linkages (designing to perform a desired motion), even though seemingly elementary, is quite difficult to perform accurately.

The difficulty really became acute during World War II. There was an obvious need for simple types of computers, in

---

* Numbers in parentheses refer to items in the Works Consulted.

central fire control systems, for example, which satisfied a given function continuously, accurately, and instantaneously for a given range of variables. While a 4-bar linkage fulfilled these needs very well, the major problem was how to design a linkage accurately.

During the war, design efforts were mostly of the graphical trial and error type, and these methods predominated until well into the 1950's. A notable example was the work by John A. Hrones and George L. Nelson (3) published in 1951. They presented a graphical series of coupler paths resulting when the cranks of given 4-bar linkages were rotated 360 degrees. Their method of design was to select one of the linkages whose resultant paths most accurately approximated the desired path. As late as 1955, Allen C. Dunk (4) published a paper in Machine Design describing a 4-bar function generator for mechanically drawing curves of required functions. His method, one of the quickest and simplest of the graphical types, employed a variance in crank length to change the character of the resulting curve.

The rapid development and use of high speed computers, however, made new analytical methods feasible. One of the first of these was given in a paper presented by B. W. Schaffer and I. Cockin (5) at the 1954 Fall meeting of the ASME. They developed a compatibility equation involving multiple derivatives of three parameters. These three parameters were (a) the direction cosine function of the crank angle of a 4-bar linkage, (b) the direction cosine function of the follower angle, and (c) the cosine function of the double angle (follower angle minus the crank angle). Using this compatibility equation, they could predict if a 4-bar linkage could be designed to satisfy a given function.

Then in 1955, Ferdinand Freudenstein (6) a foremost authority on analytical methods, presented a paper, "Approximate Synthesis of 4-Bar Linkages," at the ASME Fall meeting in which he mentioned analytical considerations in linkage synthesis.

In the Bulletin of the Virginia Polytechnic Institute for August, 1957, L. A. Padis (7) presented a method for finding the ratios between the links of a 4-bar linkage by locating the mutual rotational centers. These centers were found for a linkage in only one position, but he pointed out that additional positions could be obtained by approximating additional instant mutual centers of rotation.

In November, 1958, Freudenstein presented another paper, "4-Bar Function Generators," (8) at an ASME meeting, describing a method for linkage synthesis, employing the mathematics of complex numbers. He presented a computer program for the I.B.M. 650 which solved vector closure equations in pairs, and thereby found a mechanism that satisfied a given function precisely, for five positions.

The method of Freudenstein was of particular interest, but his system was limited to five positions by his mathematical procedure. It appeared, however, that it should be possible, by a method of iteration, to inter-relate parameters in such a way that more closure equations could be satisfied for a given linkage. The more closure equations that could be satisfied, of course, the more linkage positions that could be obtained.

This study began as an attempt to discover whether an iterative procedure would indeed work.

The first step was to derive closure equations representing a 4-bar linkage in four positions. The next step was to conduct actual trials using these closure equations with different combinations of the parameters. When it was discovered that iteration would yield a workable system for four positions, it also seemed possible that it might well work for five. After satisfactorily deriving the method for five positions, the study was concluded; it seems likely, however, that the method could be expanded to find linkages which satisfy six, and possibly seven linkage positions.

CHAPTER II

DERIVATION OF CLOSURE EQUATIONS FOR FOUR POSITIONS AND PROCEDURE OF ITERATION

Each link of a 4-bar linkage can be treated as a vector and represented by components involving complex number notation. In the 4-bar linkage below, Figure 1, this was done for the initial position of the mechanism. Closure equations were then written as the vector sum of the crank, coupler, and the follower; this sum equals the ground link (Z_1), since the mechanism is always continuous.

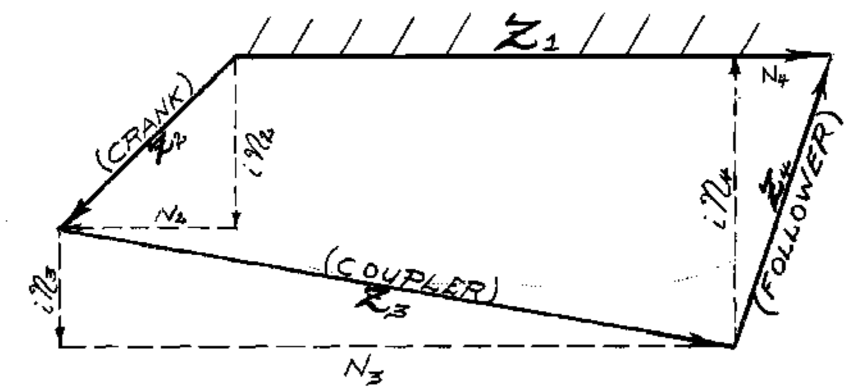

Figure 1. 4-Bar Linkage With Links Represented By Their Components.

Letting the links in Figure 1 be represented by the vectors Z_1, Z_2, Z_3, and Z_4, the first general closure equation is written for position 1:

$$\text{Position } \underline{1}: \quad Z_2 + Z_3 + Z_4 = Z_1 \qquad \text{(Eqn 1)}$$

Figure 2 illustrates the 4-bar linkage in its initial and three subsequent positions. Counter-clockwise rotation is assumed positive. The coupler is shown in its initial position only.

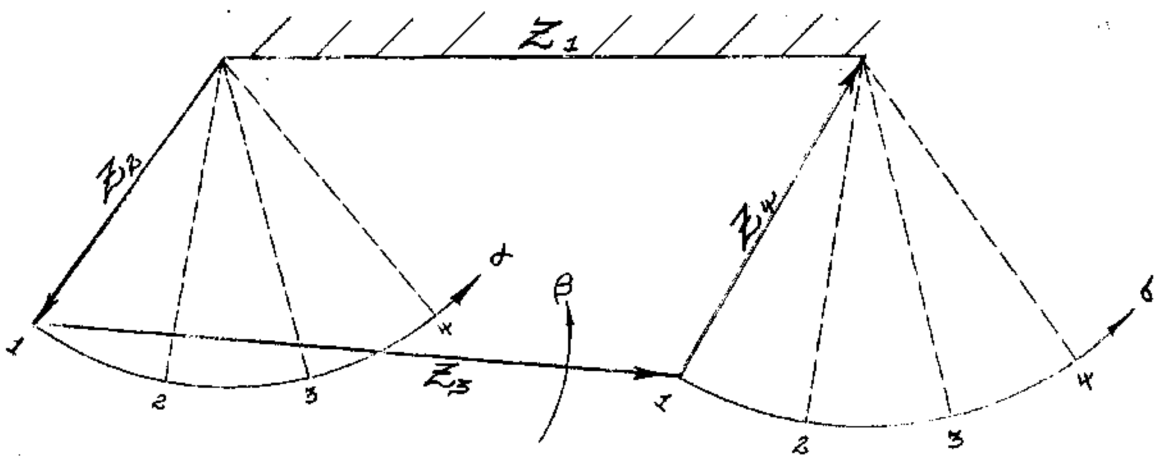

Figure 2. Four Positions For A 4-Bar Linkage

From Figure 2 three additional closure equations were written representing the three rotated positions 2, 3, and 4. The angle of rotation of the crank is α degrees, of the coupler β degrees, and of the follower σ degrees. All of these angles were measured from initial positions to rotated positions. These additional closure equations are:

Rotation to position 2: $R\alpha_{12}Z_2 + R\beta_{12}Z_3 + R\sigma_{12}Z_4 = Z_1$ (Eqn 2)

Rotation to position 3: $R\alpha_{13}Z_2 + R\beta_{13}Z_3 + R\sigma_{13}Z_4 = Z_1$ (Eqn 3)

Rotation to position 4: $R\alpha_{14}Z_2 + R\beta_{14}Z_3 + R\sigma_{14}Z_4 = Z_1$ (Eqn 4)

Where $R_\phi = \cos \phi + i \sin \phi$

Subtracting equation 1 from each of equations 2, 3, and 4, the following was obtained:

$$(R\alpha_{12}-1)Z_2 + (R\beta_{12}-1)Z_3 + (R\sigma_{12}-1)Z_4 = 0 \qquad \text{(Eqn 5)}$$

$$(R\alpha_{13}-1)Z_2 + (R\beta_{13}-1)Z_3 + (R\sigma_{13}-1)Z_4 = 0 \qquad \text{(Eqn 6)}$$

$$(R\alpha_{14}-1)Z_2 + (R\beta_{14}-1)Z_3 + (R\sigma_{14}-1)Z_4 = 0 \qquad \text{(Eqn 7)}$$

This subtraction eliminates the ground link Z_1, and the mechanism is thus completely defined by its moving links. Substituting the identity $(R_\phi-1) = \lambda_\phi$, in equation 5, 6, and 7, the following forms of the closure equations were obtained:

$$\lambda\alpha_{12}Z_2 + \lambda\beta_{12}Z_3 + \lambda\sigma_{12}Z_4 = 0 \qquad \text{(Eqn 8)}$$

$$\lambda\alpha_{13}Z_2 + \lambda\beta_{13}Z_3 + \lambda\sigma_{13}Z_4 = 0 \qquad \text{(Eqn 9)}$$

$$\lambda\alpha_{14}Z_2 + \lambda\beta_{14}Z_3 + \lambda\sigma_{14}Z_4 = 0 \qquad \text{(Eqn 10)}$$

It is possible to solve these equations simultaneously for the unknowns, Z_2, Z_3, and Z_4; however, it is more convenient to express links as ratios among themselves. The coupler (Z_3) is therefore assumed unity and equations 8, 9, and 10 rewritten:

$$\lambda\alpha_{12}Z_2 + \lambda\beta_{12}\cdot 1 + \lambda\sigma_{12}Z_4 = 0 \qquad \text{(Eqn 11)}$$

$$\lambda\alpha_{13}Z_2 + \lambda\beta_{13}\cdot 1 + \lambda\sigma_{13}Z_4 = 0 \qquad \text{(Eqn 12)}$$

$$\lambda\alpha_{14}Z_2 + \lambda\beta_{14}\cdot 1 + \lambda\sigma_{14}Z_4 = 0 \qquad \text{(Eqn 13)}$$

The unknown links in these equations are Z_2 and Z_4, which could be found by solving equations 11 and 12 simultaneously; but the resulting 4-bar mechanism satisfied only three positions. When any one of the full ranges of rotation (α_{14}, β_{14}, or σ_{14}, in equation 13) is considered a variable, and any variance is reflected back into equations 11 and 12, then equation 13 can impose an additional position on the solution for Z_2 and Z_4. This implies that α_{12}, α_{13}; β_{12}, β_{13}; and σ_{12}, σ_{13} be proportional parts of α_{14}, β_{14}, and σ_{14} respectively.

If a solution for Z_2 and Z_4 satisfying all three of the closure equations does exist, the following determinant must equal zero.

$$\begin{vmatrix} \lambda_{\alpha_{12}} & \lambda_{\beta_{12}} & \lambda_{\sigma_{12}} \\ \lambda_{\alpha_{13}} & \lambda_{\beta_{13}} & \lambda_{\sigma_{13}} \\ \lambda_{\alpha_{14}} & \lambda_{\beta_{14}} & \lambda_{\sigma_{14}} \end{vmatrix} = 0$$

Therefore, if α_{12} and α_{13} are proportional parts of α_{14}; β_{12} and β_{13} of β_{14}; σ_{12} and σ_{13} of σ_{14}, then one of the full ranges (α_{14}, β_{14}, or σ_{14}) could be determined (the other two remaining constant) that will cause the determinant to equal zero. If this value were substituted into equation 13 and then reflected into equations 11 and 12, the resulting solution for links Z_2 and Z_4 would satisfy all three equations.

The solution of the determinant could be found directly and the value used to solve for Z_2 and Z_4, but this method becomes so involved an iteration process seems more efficient. This iteration was performed by assuming ways of reflecting the ranges back into the rotations in equations 11 and 12, solving for links Z_2 and Z_4 from equations 11 and 12, and then varying one of the full ranges of rotation in an effort to cause Z_2 and Z_4 to satisfy equation 13. (Equations 11 and 12 are always satisfied by the resulting Z_2 and Z_4). Thus, by iteration, a mechanism and a range of rotation value* are found which satisfy all three closure equations.

---

*This value also satisfies the determinant shown on page 9.

CHAPTER III

DETERMINATION OF A SATISFACTORY INTER-RELATION OF ROTATIONAL PARAMETERS

The general iteration procedure shown in the preceding chapter was used as a tool in an attempt to find a satisfactory inter-relation between the rotational parameters. It was desired to find which full range of rotation (in equation 13) could be varied most satisfactorily and how it might best be reflected back into the first two closure equations (11 and 12). The mechanics of the iterative trials were thus conducted as follows:

α_{14}, β_{14}, and σ_{14} were first assumed; α_{12}, α_{13} and β_{12}, β_{13} were assumed to be some proportions of α_{14} and β_{14} respectively. With these assumed values, the only remaining unknowns in the closure equations were Z_2 and Z_4. Equations 11 and 12 were then solved by Cramer's rules, using second order determinants.

Writing Z_2 as a determinant from equations 11 and 12 gives:

$$Z_2 = \frac{\begin{vmatrix} (-\lambda_{\beta_{12}}) & (\lambda_{\sigma_{12}}) \\ (-\lambda_{\beta_{13}}) & (\lambda_{\sigma_{13}}) \end{vmatrix}}{\begin{vmatrix} (\lambda_{\alpha_{12}}) & (\lambda_{\sigma_{12}}) \\ (\lambda_{\alpha_{13}}) & (\lambda_{\sigma_{13}}) \end{vmatrix}}$$

$$Z_2 = \frac{(-\lambda_{\beta_{12}})(\lambda_{\sigma_{13}}) - (\lambda_{\sigma_{12}})(-\lambda_{\beta_{13}})}{(\lambda_{\alpha_{12}})(\lambda_{\sigma_{13}}) - (\lambda_{\sigma_{12}})(\lambda_{\alpha_{13}})}$$

Or rewriting:

$$Z_2 = \frac{(\lambda_{\sigma_{12}})(\lambda_{\beta_{13}}) - (\lambda_{\beta_{12}})(\lambda_{\sigma_{13}})}{(\lambda_{\alpha_{12}})(\lambda_{\sigma_{13}}) - (\lambda_{\sigma_{12}})(\lambda_{\alpha_{13}})} \qquad \textbf{(Eqn 14)}$$

Substituting $\lambda_\phi = (\cos\phi - 1) + i\sin\phi$ **, the expression for** Z_2 **is further expanded:**

(Eqn 15)

$$\frac{[(\cos\sigma_{12}-1)+i\sin\sigma_{12}][(\cos\beta_{13}-1)+i\sin\beta_{13}] - [(\cos\beta_{12}-1)+i\sin\beta_{12}][(\cos\sigma_{13}-1)+i\sin\sigma_{13}]}{[(\cos\alpha_{12}-1)+i\sin\alpha_{12}][(\cos\sigma_{13}-1)+i\sin\sigma_{13}] - [(\cos\sigma_{12}-1)+i\sin\sigma_{12}][(\cos\alpha_{13}-1)+i\sin\alpha_{13}]}$$

Similarly Z_4 **is written as a determinant from equations <u>11</u> and <u>12</u> as follows:**

$$Z_4 = \frac{\begin{vmatrix} (\lambda_{\alpha_{12}}) & (-\lambda_{\beta_{12}}) \\ (\lambda_{\alpha_{13}}) & (-\lambda_{\beta_{13}}) \end{vmatrix}}{\begin{vmatrix} (\lambda_{\alpha_{12}}) & (\lambda_{\sigma_{12}}) \\ (\lambda_{\alpha_{13}}) & (\lambda_{\sigma_{13}}) \end{vmatrix}}$$

$$Z_4 = \frac{(\lambda_{\alpha_{12}})(-\lambda_{\beta_{13}}) - (-\lambda_{\beta_{12}})(\lambda_{\alpha_{13}})}{(\lambda_{\alpha_{12}})(\lambda_{\sigma_{13}}) - (\lambda_{\sigma_{12}})(\lambda_{\alpha_{13}})}$$

Or rewriting in the form:

$$Z_4 = \frac{(\lambda_{\beta_{12}})(\lambda_{\alpha_{13}}) - (\lambda_{\alpha_{12}})(\lambda_{\beta_{13}})}{(\lambda_{\alpha_{12}})(\lambda_{\sigma_{13}}) - (\lambda_{\sigma_{12}})(\lambda_{\alpha_{13}})} \qquad \text{(Eqn 16)}$$

Again, making the substitution $\lambda_\phi = (\cos\phi - 1) + i\sin\phi$, this expression for Z_4 is further expanded:

(Eqn 17)

$$\frac{[(\cos\beta_{12}-1)+i\sin\beta_{12}][(\cos\alpha_{13}-1)+i\sin\alpha_{13}]-[(\cos\alpha_{12}-1)+i\sin\alpha_{12}][(\cos\beta_{13}-1)+i\sin\beta_{13}]}{[(\cos\alpha_{12}-1)+i\sin\alpha_{12}][(\cos\sigma_{13}-1)+i\sin\sigma_{13}]-[(\cos\sigma_{12}-1)+i\sin\sigma_{12}][(\cos\alpha_{13}-1)+i\sin\alpha_{13}]}$$

Equations <u>14</u> and <u>16</u> were initially expanded using the identity $\lambda_\phi \lambda_\psi = \lambda_{\phi+\psi} - \lambda_\phi - \lambda_\psi$, but it was found that a large inaccuracy frequently resulted due to the subtraction of "nearly like" numbers, sometimes reducing the remainder to as few as two significant figures. Obviously when these remainders were then further operated on (multiplied, divided, etc.), the final resulting solutions for Z_2 and Z_4 represented a large accumulative error. It was found when the resulting mechanism was plotted and rotated, it did not satisfy the first and second rotations exactly. Several times, an error in excess of 50% was observed. Therefore, equations <u>14</u> and <u>16</u> were expanded as shown into equations <u>15</u> and <u>17</u> and the multiplication performed.

After the initial multiplication, the numerator and the denominator were multiplied by the conjugate of the denominator. The denominator was reduced to a single term by addition, and divided into each term in the numerator. Then the resulting like terms were collected and the result became the solution for Z_2 or Z_4. The result took the form:

$$Z = N + i\eta$$

Where N is the real (horizontal) component of the link, and $i\eta$ is the imaginary (vertical) component of the link.

At this stage in the development, an I.B.M. 650 computer program was written and employed in the simultaneous solution of equations 15 and 17 for links Z_2 and Z_4. This program for the four positions case is shown in Appendix A.

The resulting mechanisms determined by Z_2 and Z_4 were constructed graphically and the crank rotated the assumed α_{14} degrees. Rotation of the follower was then measured and compared with desired σ_{14}, appearing in the third closure equation, 13.* Obviously, if the angle of rotation of the follower, resulting from the α_{14} crank rotation was not equal to assumed σ_{14}, then equation 13 was not satisfied. Thus, one of the assumed ranges and/or the proportions of this range, which reflects rotation (1-4) back into rotations (1-2) and (1-3), was varied in an

---

*Of course, all rotations in equations 11 and 12 were always satisfied by the resulting mechanism.

effort to satisfy equation 13 by causing the resulting σ_{14} to approach the assumed σ_{14}. In this way, the iteration process was conducted for many different trials in an effort to determine which range was best varied, and how it was best reflected back into its subranges of rotations. The "best" method was determined by actually trying different combinations of the parameters and analyzing the results graphically to see which yielded the most workable mechanism.

To avoid basing a conclusion on only one function, three, and in some cases four, different functions were mechanized for each different trial. The following table was constructed for the nine trials conducted, showing assumed variables for full ranges of rotation, the assumed functions of the sub-rotations, and the corresponding results.

Table 1. Summary of Trials

Trial No.	Rotational Ranges		Sub-Rotation Functions		Results
	Variables	Constants	Crank	Coupler	
1	σ_{14}	α_{14}, β_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=\frac{1}{3}\beta_{14}$ $\beta_{13}=\frac{2}{3}\beta_{14}$	No converging trend was observed for the normally assumed values of the variable range.
2	σ_{14} $\beta_{14} \propto \sigma_{14}$	α_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=\frac{1}{3}\beta_{14}$ $\beta_{13}=\frac{2}{3}\beta_{14}$	The resultant follower rotations (σ_{14}) were virtually the same as those observed in Trial #1, and no converging trend was indicated.
3	σ_{14}	α_{14}, β_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=f^{*}(\beta_{14})$ $\beta_{13}=f^{*}(\beta_{14})$	A definite diverging trend was observed for the resulting values of ${}_{14}$ in three of the four equations mechanized. The fourth equation resulted in a mechanism in which the crank could not be rotated the desired full range α_{14}.

*This function was given by the equation being mechanized.

Table 1. Summary of Trials (Continued)

Trial No.	Rotational Ranges		Sub-Rotation Functions		Results
	Variables	Constants	Crank	Coupler	
4	σ_{14} $\beta_{14} \propto \sigma_{14}$	α_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=f^*(\beta_{14})$ $\beta_{13}=f^*(\beta_{14})$	The additional condition of letting β_{14} vary proportionally with σ_{14}, had no apparent effect in causing the resulting values of σ_{14} to be any different from those observed in Trial #3. Therefore, the resulting values of σ_{14} again diverged.
5	---	α_{14} β_{14} σ_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=f^*(\beta_{14})$ $\beta_{13}=f^*(\beta_{14})$	This trial was conducted to investigate how these functions alone for the crank and coupler affected the resulting value of σ_{14}. The resulting mechanisms were so out of proportion that it was impossible to graphically construct them. In one case the coupler was 1200 times larger than the crank.

*This function was given by the equation being mechanized.

Table 1. Summary of Trials (Continued)

Trial No.	Rotational Ranges		Sub-Rotation Functions		Results
	Variables	Constants	Crank	Coupler	
6	α_{14}	β_{14}, σ_{14}	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=\frac{1}{3}\beta_{14}$ $\beta_{13}=\frac{2}{3}\beta_{14}$	Large variations in the ranges of the cranks resulted in extremely small variations in the resultant ranges of the followers.
7	α_{14}	β_{14}, σ_{14}	$\alpha_{12}=f^*(\alpha_{14})$ $\alpha_{13}=f^*(\alpha_{14})$	$\beta_{12}=\frac{1}{3}\beta_{14}$ $\beta_{13}=\frac{2}{3}\beta_{14}$	The cranks and followers of all the resultant mechanisms were several hundred times larger than their respective couplers.
8	α_{14}	β_{14}, σ_{14}	$\alpha_{12}=f^*(\alpha_{14})$ $\alpha_{13}=f^*(\alpha_{14})$	$\beta_{12}=f^*(\beta_{14})$ $\beta_{13}=f^*(\beta_{14})$	Because the sub-rotations for all three moving links were the same function of their respective ranges, all resultant mechanisms bordered on "parallelogramism." This led to unproportioned mechanisms, which were difficult to construct graphically. Therefore, no conclusions were reached concerning convergence or divergence of the resulting rotations of the follower. It was interesting to note that when $\alpha_{14}=\sigma_{14}$, and $\beta_{14}=0$, a parallelogram resulted.

*This function was given by the equation being mechanized.

Table 1. Summary of Trials (Concluded)

Trial No.	Rotational Ranges		Sub-Rotation Functions		Results
	Variables	Constants	Crank	Coupler	
9	β_{14}	α_{14}, $\tilde{\sigma}_{14}$	$\alpha_{12}=\frac{1}{3}\alpha_{14}$ $\alpha_{13}=\frac{2}{3}\alpha_{14}$	$\beta_{12}=\frac{1}{3}\beta_{14}$ $\beta_{13}=\frac{2}{3}\beta_{14}$	The systematic variance of the range of the coupler caused the resultant follower rotation ($\tilde{\sigma}_{14}$) to converge upon the desired value of $\tilde{\sigma}_{14}$. The resulting mechanisms were well proportioned, enabling graphical constructions to be easily made. This trial was included in the appendix to show details of how the convergence of the resulting $\tilde{\sigma}_{14}$ was determined.

Conclusions.--Certainly many more combinations are possible; however, those presented seem most obvious. Thus, when a satisfactory inter-relation of parameters was found in the ninth trial, the series was concluded. The methods by which all nine trials were conducted are very similar; the details of trial 9 are included in Appendix B to illustrate the method.

The graphical analysis of the resulting σ_{14} was not sufficiently accurate to enable the results of a small interval of variance of β_{14} to be observed. For example, in trial 9, it was found that some value between 30.0 and 36.0 degrees for β_{14} would cause the resulting σ_{14} to approach the desired value of σ_{14}; finding this value of β_{14} with any accuracy was difficult though because the graphical analysis of σ_{14} easily contained an error of $\pm$ 0.2 degrees, even though the mechanisms were plotted to a very large scale. Thus, it became necessary to analyze the resulting σ_{14} by analytical means. It was observed, however, during the derivation of an analytical method for the analysis of σ_{14}, that it would be possible to expand this iterative type of system to include more than four positions by introducing additional variables, and writing more closure equations. Thus, closure equations were derived for five positions by including the coupler as a variable and using the combination of parameters found in trial 9. The analytical method for determining the resultant follower rotation was, therefore, derived for five positions instead of four. These derivations are shown in the next chapter.

CHAPTER IV

DERIVATION OF CLOSURE EQUATIONS AND ANALYTICAL METHOD FOR DETERMINING RESULTANT FOLLOWER ROTATION FOR FIVE POSITIONS

Derivation of the closure equations for five positions was similar to that for four positions, in that it also involved complex numbers, which represented vectors and the complex rotational methods. The derivation was begun by drawing a 4-bar linkage in five positions.

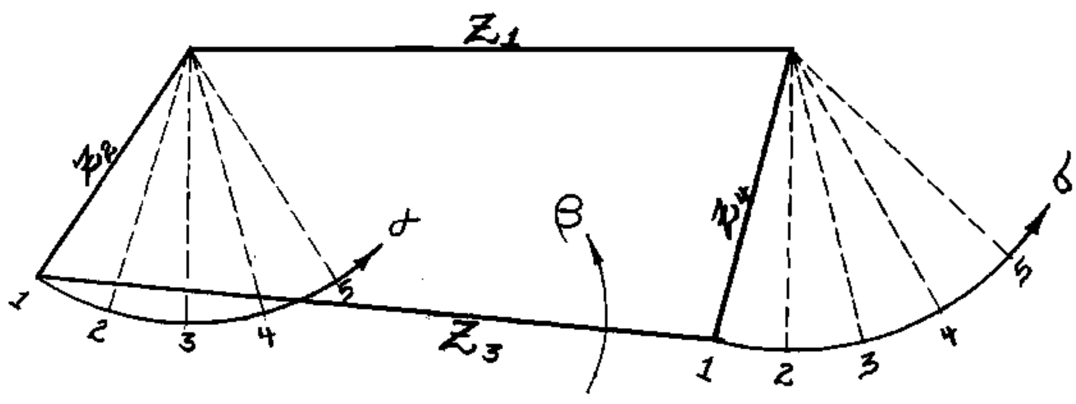

Figure 3. 4-Bar Linkage Shown in Five Positions.

For clarity, in Fig. 3, the coupler is shown in its initial position only. All rotations are positive, counter-clockwise as shown.

Vector equations for the mechanism in Fig. 3, in its initial and rotated positions, were written:

Position

1 $$Z_2 + Z_3 + Z_4 = Z_1 \quad \text{(Eqn 18)}$$

2 $$R\alpha_{12}Z_2 + R\beta_{12}Z_3 + R\sigma_{12}Z_4 = Z_1 \quad \text{(Eqn 19)}$$

3 $$R\alpha_{13}Z_2 + R\beta_{13}Z_3 + R\sigma_{13}Z_4 = Z_1 \quad \text{(Eqn 20)}$$

4 $$R\alpha_{14}Z_2 + R\beta_{14}Z_3 + R\sigma_{14}Z_4 = Z_1 \quad \text{(Eqn 21)}$$

5 $$R\alpha_{15}Z_2 + R\beta_{15}Z_3 + R\sigma_{15}Z_4 = Z_1 \quad \text{(Eqn 22)}$$

Instead of eliminating the ground link (Z_1) by subtracting equations as in the derivation for four positions, Z_1 was assumed unity, with the coupler considered a variable. $Z_1 = 1.0 + i0.0$, equations 19 20, and 21 were solved simultaneously for the three unknowns Z_2, Z_3, and Z_4. These solutions are actually ratios to unity, due to the assumed value for Z_1. The method of varying the range of the coupler, as found in trial 9, was then used to satisfy equation 22 by iteration.

From equations 19, 20, and 21 the solution for the crank (Z_2) is expressed as:

$$
\mathcal{Z}_2 = \frac{\begin{vmatrix} \mathcal{Z}_1 & \mathcal{R}_{\beta_{12}} & \mathcal{R}_{\sigma_{12}} \\ \mathcal{Z}_1 & \mathcal{R}_{\beta_{13}} & \mathcal{R}_{\sigma_{13}} \\ \mathcal{Z}_1 & \mathcal{R}_{\beta_{14}} & \mathcal{R}_{\sigma_{14}} \end{vmatrix}}{\begin{vmatrix} \mathcal{R}_{\alpha_{12}} & \mathcal{R}_{\beta_{12}} & \mathcal{R}_{\sigma_{12}} \\ \mathcal{R}_{\alpha_{13}} & \mathcal{R}_{\beta_{13}} & \mathcal{R}_{\sigma_{13}} \\ \mathcal{R}_{\alpha_{14}} & \mathcal{R}_{\beta_{14}} & \mathcal{R}_{\sigma_{14}} \end{vmatrix}}
$$

Expanding and grouping terms gives:

$$
\frac{\mathcal{Z}_1(\mathcal{R}_{\beta_{13}}\mathcal{R}_{\sigma_{14}} + \mathcal{R}_{\beta_{12}}\mathcal{R}_{\sigma_{13}} + \mathcal{R}_{\beta_{14}}\mathcal{R}_{\sigma_{12}}) - \mathcal{Z}_1(\mathcal{R}_{\sigma_{12}}\mathcal{R}_{\beta_{13}} + \mathcal{R}_{\sigma_{13}}\mathcal{R}_{\beta_{14}} + \mathcal{R}_{\beta_{12}}\mathcal{R}_{\sigma_{14}})}{(\mathcal{R}_{\alpha_{12}}\mathcal{R}_{\beta_{13}}\mathcal{R}_{\sigma_{14}} + \mathcal{R}_{\alpha_{13}}\mathcal{R}_{\beta_{14}}\mathcal{R}_{\sigma_{12}} + \mathcal{R}_{\beta_{12}}\mathcal{R}_{\sigma_{13}}\mathcal{R}_{\alpha_{14}}) - (\mathcal{R}_{\sigma_{12}}\mathcal{R}_{\beta_{13}}\mathcal{R}_{\alpha_{14}} + \mathcal{R}_{\beta_{12}}\mathcal{R}_{\alpha_{13}}\mathcal{R}_{\sigma_{14}} + \mathcal{R}_{\alpha_{12}}\mathcal{R}_{\beta_{14}}\mathcal{R}_{\sigma_{13}})}
$$

The numerator was expanded further by substituting unity for Z_1, and using the rotational identity $\mathcal{R}_\phi = \cos\phi + i\sin\phi$ **Since the denominator was the same for the determinants of all three links, it was of course only expanded and evaluated once. This is shown on page 27, following the expansion for the follower (Z_4).**

$$\mathcal{Z}_2 = \qquad \text{(Eqn 23)}$$

$$\left[\frac{1.0\left[(\cos\beta_{13} + i\sin\beta_{13})(\cos\sigma_{14} + i\sin\sigma_{14}) + (\cos\beta_{12} + i\sin\beta_{12})(\cos\sigma_{13} + i\sin\sigma_{13})\right.}{DENOM} + \frac{\left.(\cos\beta_{14} + i\sin\beta_{14})(\cos\sigma_{12} + i\sin\sigma_{12})\right]}{DENOM}\right]$$

$$-\left[\frac{1.0\left[(\cos\sigma_{12} + i\sin\sigma_{12})(\cos\beta_{13} + i\sin\beta_{13}) + (\cos\sigma_{13} + i\sin\sigma_{13})(\cos\beta_{14} + i\sin\beta_{14})\right.}{DENOM} + \frac{\left.(\cos\beta_{12} + i\sin\beta_{12})(\cos\sigma_{14} + i\sin\sigma_{14})\right]}{DENOM}\right]$$

Now, in the same manner, the determinant for the coupler (Z_3) is written:

$$\mathcal{Z}_3 = \frac{\begin{vmatrix} R_{\alpha_{12}} & \mathcal{Z}_1 & R_{\sigma_{12}} \\ R_{\alpha_{13}} & \mathcal{Z}_1 & R_{\sigma_{13}} \\ R_{\alpha_{14}} & \mathcal{Z}_1 & R_{\sigma_{14}} \end{vmatrix}}{\left| DENOM \right|}$$

The numerator of this determinant is expanded further by substituting unity for Z_1, and using the rotational identity $R_\phi = \cos\phi + i\sin\phi$.

$$\frac{Z_1\left(R_{\alpha_{12}} R_{\sigma_{14}} + R_{\sigma_{13}} R_{\alpha_{14}} + R_{\alpha_{13}} R_{\sigma_{12}}\right) - Z_1\left(R_{\sigma_{12}} R_{\alpha_{14}} + R_{\sigma_{13}} R_{\alpha_{12}} + R_{\alpha_{13}} R_{\sigma_{14}}\right)}{DENOM}$$

$$Z_3 = \qquad \text{(Eqn 24)}$$

$$\left[\frac{1.0\left[(\cos\alpha_{12} + i\sin\alpha_{12})(\cos\sigma_{14} + i\sin\sigma_{14}) + (\cos\sigma_{13} + i\sin\sigma_{13})(\cos\alpha_{14} + i\sin\alpha_{14})\right.}{DENOM} + \frac{\left.(\cos\alpha_{13} + i\sin\alpha_{13})(\cos\sigma_{12} + i\sin\sigma_{12})\right]}{DENOM}\right]$$

$$-\left[\frac{1.0\left[(\cos\sigma_{12} + i\sin\sigma_{12})(\cos\alpha_{14} + i\sin\alpha_{14}) + (\cos\sigma_{13} + i\sin\sigma_{13})(\cos\alpha_{12} + i\sin\alpha_{12})\right.}{DENOM} + \frac{\left.(\cos\alpha_{13} + i\sin\alpha_{13})(\cos\sigma_{14} + i\sin\sigma_{14})\right]}{DENOM}\right]$$

Similarly, a determinant for the solution of the follower (Z_4) is written:

$$Z_4 = \frac{\begin{vmatrix} R\alpha_{12} & R\beta_{12} & Z_1 \\ R\alpha_{13} & R\beta_{13} & Z_1 \\ R\alpha_{14} & R\beta_{14} & Z_1 \end{vmatrix}}{\begin{vmatrix} \\ DENOM \\ \\ \end{vmatrix}}$$

Expanding and grouping terms gives:

$$\frac{Z_1(R\alpha_{12}R\beta_{13} + R\beta_{12}R\alpha_{14} + R\alpha_{13}R\beta_{14}) - Z_1(R\beta_{13}R\alpha_{14} + R\beta_{14}R\alpha_{12} + R\beta_{12}R\alpha_{13})}{DENOM}$$

Letting $Z_1 = 1.0 + i0.0$ and substituting the identity

$$R_\phi = \cos\phi + i\sin\phi :$$

$$\mathcal{Z}_4 = \qquad \text{(Eqn 25)}$$

$$\left[\frac{1.0\left[(\cos\alpha_{12}+i\sin\alpha_{12})(\cos\beta_{13}+i\sin\beta_{13})+(\cos\beta_{12}+i\sin\beta_{12})(\cos\alpha_{14}+i\sin\alpha_{14})\right.}{DENOM}\right.$$

$$\left.+\frac{\left.(\cos\alpha_{13}+i\sin\alpha_{13})(\cos\beta_{14}+i\sin\beta_{14})\right]}{DENOM}\right]$$

$$-\left[\frac{1.0\left[(\cos\beta_{13}+i\sin\beta_{13})(\cos\alpha_{14}+i\sin\alpha_{14})+(\cos\beta_{14}+i\sin\beta_{14})(\cos\alpha_{12}+i\sin\alpha_{12})\right.}{DENOM}\right.$$

$$\left.+\frac{\left.(\cos\beta_{12}+i\sin\beta_{12})(\cos\alpha_{13}+i\sin\alpha_{13})\right]}{DENOM}\right]$$

Finally, the denominator, which is the same for all three links Z_2, Z_3, and Z_4, was evaluated. The denominator has already been shown in a determinant and in the expanded form, in conjunction with the crank (Z_2), on page 22, but is repeated as follows for the sake of continuity.

The denominator:

$$\overline{\begin{vmatrix} R_{\alpha_{12}} & R_{\beta_{12}} & R_{\sigma'_{12}} \\ R_{\alpha_{13}} & R_{\beta_{13}} & R_{\sigma'_{13}} \\ R_{\alpha_{14}} & R_{\beta_{14}} & R_{\sigma'_{14}} \end{vmatrix}}$$

Expanding and grouping terms gives:

$$\overline{\left(R_{\alpha_{12}} R_{\beta_{13}} R_{\sigma'_{14}} + R_{\alpha_{13}} R_{\beta_{14}} R_{\sigma'_{12}} + R_{\beta_{12}} R_{\sigma'_{13}} R_{\alpha_{14}}\right) - \left(R_{\sigma'_{12}} R_{\beta_{13}} R_{\alpha_{14}} + R_{\beta_{12}} R_{\alpha_{13}} R_{\sigma'_{14}} + R_{\alpha_{12}} R_{\beta_{14}} R_{\sigma'_{13}}\right)}$$

Substituting the identity $R_\phi = \cos\phi + i\sin\phi$ **:**

$$\Big[\overline{(\cos\alpha_{12} + i\sin\alpha_{12})(\cos\beta_{13} + i\sin\beta_{13})(\cos\sigma'_{14} + i\sin\sigma'_{14})} \cdots$$

$$\cdots + \overline{(\cos\alpha_{13} + i\sin\alpha_{13})(\cos\beta_{14} + i\sin\beta_{14})(\cos\sigma'_{12} + i\sin\sigma'_{12})} \cdots$$

$$\cdots + \overline{(\cos\beta_{12} + i\sin\beta_{12})(\cos\sigma'_{13} + i\sin\sigma'_{13})(\cos\alpha_{14} + i\sin\alpha_{14})} \cdots$$

$$\cdots - \overline{(\cos\sigma'_{12} + i\sin\sigma'_{12})(\cos\beta_{13} + i\sin\beta_{13})(\cos\alpha_{14} + i\sin\alpha_{14})} \cdots$$

$$\cdots \frac{}{(\cos\beta_{12} + i\sin\beta_{12})(\cos\alpha_{13} + i\sin\alpha_{13})(\cos\sigma_{14} + i\sin\sigma_{14})} \cdots$$

$$\left. \cdots \frac{}{(\cos\alpha_{12} + i\sin\alpha_{12})(\cos\beta_{14} + i\sin\beta_{14})(\cos\sigma_{13} + i\sin\sigma_{13})} \right]$$

At this point an I.B.M. 650 computer program was written for the solutions of equations 23, 24, and 25. This program for five positions is shown in Appendix C.

Analysis of the resulting range of follower rotation (σ_{15}), when the crank was rotated its assumed full range (α_{15}), was previously performed by graphically constructing the resulting mechanism and actually measuring the rotation of the follower. However, due to the inaccuracies of this method, an analytical method was derived to determine this resultant follower rotation. This method enabled the iteration of the coupler range to be performed with great precision, because small variations in the resulting follower range could be observed.

Derivation of Analytical Method For Determining Resulting Follower Rotation For Five Positions.--For this derivation, equations 18 and 22 are repeated:

$$Z_2 + \quad Z_3 + \quad Z_4 = Z_1 \qquad \text{(Eqn 18)}$$

$$R\alpha_{15}Z_2 + R\beta_{15}Z_3 + R\sigma_{15}Z_4 = Z_1 \qquad \text{(Eqn 22)}$$

Subtracting equation 18 from equation 22 gives:

$$(R\alpha_{15}-1)Z_2 + (R\beta_{15}-1)Z_3 + (R\sigma_{15}-1)Z_4 = 0$$

And by substituting the identity $(R_\phi - 1) = \lambda_\phi$, where $\lambda_\phi = (\cos\phi - 1) + i\sin\phi$, the result is:

$$\lambda\alpha_{15}Z_2 + \lambda\beta_{15}Z_3 + \lambda\sigma_{15}Z_4 = 0 \qquad \text{(Eqn 26)}$$

Solving equation 26 $\lambda_{\sigma_{15}}$ yields:

$$\lambda_{\sigma_{15}} = \frac{-\lambda\alpha_{15}Z_2 - \lambda\beta_{15}Z_3}{Z_4}$$

Letting the components of each link be represented by $Z_a = N_a + i\eta_a$, $(a = 2, 3, 4)$, and substituting the above identity for $\lambda\alpha_{15}$ and $\lambda\beta_{15}$, the result is:

(Eqn 27)

$$\lambda_{\sigma_{15}} = \frac{-[(\cos\alpha_{15} - 1) + i\sin\alpha_{15}][N_2 + i\eta_2] - [(\cos\beta_{15} - 1) + i\sin\beta_{15}][N_3 + i\eta_3]}{[N_4 + i\,\eta_4]}$$

The numerator and the denominator are multiplied by the conjugate of the denominator. The fraction is then reduced by division and the result takes the form:

$$\lambda_{\sigma_{15}} = N + i\eta$$

Substituting the subtractional rotation identity for $\lambda\sigma_{15}$:

$$\lambda\sigma_{15} = (\cos\sigma_{15} - 1) + i\sin\sigma_{15} = N + i\eta$$

Grouping related terms:

$$\cos\sigma_{15} = N + 1 \quad ; \quad i\sin\sigma_{15} = i\eta$$

Dividing imaginary term by real term:

$$\frac{i\sin\sigma_{15}}{\cos\sigma_{15}} = \frac{i\,\eta}{N + 1} = \tan\sigma_{15}$$

So, the final expression for σ_{15} is:

$$\sigma_{15} = \arctan\left(\frac{\eta}{N + 1}\right)$$

This analytical method was programmed for computer solution and is included in Appendix C as part of the method for five positions.

Thus, each resulting mechanism, (Z_2, Z_3, Z_4), along with values for the ranges of the crank and the coupler, were substituted into equation 27 to find the value of follower rotation.

Being able to determine this resultant value by computer means makes it possible to conduct the entire iteration method accurately and quickly in one computer session. This computer iteration process is described in the following chapter.

CHAPTER V

ITERATION PROCESS WITH THE I.B.M. 650 COMPUTER

The following is a brief description of the iteration method as performed on the computer. Steps 1 through 3 were performed by the operator and entered as data cards to the computer. The actual computer operation began with step 4.

1. Values for α_{15}, β_{15}, and σ_{15} were assumed.
2. The sub-ranges for the crank and the coupler are determined by making equal divisions of α_{15} and β_{15}.
3. The sub-ranges for the follower were determined by the function to be generated and the assumed value for σ_{15}.
4. With these sub-ranges known, equations 23, 24, and 25 were solved simultaneously for Z_2, Z_3, and Z_4.
5. Results for Z_2, Z_3, and Z_4, along with the assumed values for α_{15} and β_{15}, were then substituted into equation 27 and the resulting value for σ_{15} was found.
6. The resulting value of σ_{15} was compared with the

desired (originally assumed) value of σ'_{15}. This comparison was made by subtracting the desired σ'_{15} from the resulting σ'_{15}. If the result was plus, the originally assumed range of the coupler (β_{15}) was reduced by 4.0 and its sub-ranges reduced proportionally.

7. Using these new values for the coupler rotation, Z_2, Z_3, and Z_4 were again solved for.
8. The resulting σ'_{15} was determined again as in step 5.
9. It was hoped that this second resulting value of σ'_{15} would be closer to the desired value of σ'_{15} than before. If it were, the equation being mechanized was such that a decrease in the range of the coupler caused a decrease in the resultant range of the follower. If the equation being mechanized is such that a decrease in the range of the coupler causes an increase in the resultant range of the follower, then one program card must be changed to cause the iteration to proceed in the right direction. (This is explained completely in the outline in Chapter VII.) When the resulting value of σ'_{15} was closer to the desired value, the iteration was allowed to proceed. The computer further reduced by 4.0,

each time, the rotational value (β_{15}) of the coupler. Finally the resulting σ'_{15} becomes less than the desired σ'_{15}. Then the last values by which the coupler ranges were reduced were added back, and the iteration again continued; however, now the ranges of rotation of the coupler were reduced by 1/10 of what they were before. Therefore, the resulting σ'_{15} now approached more slowly the desired σ'_{15}.

10. This loop consisted of (a) reducing the values of the rotations of the coupler and causing the resulting σ'_{15} to become less than the desired σ'_{15}, (b) adding this interval back to the coupler's rotations, and (c) continuing the iteration with smaller values for reducing the coupler's rotations. This loop was made four times.
11. When the computer completed four loops, the final resulting mechanism was given, along with the final value of β_{15}.

This process was completely automatic and, once begun, the operator had only to observe the first two resulting values for σ'_{15} to ascertain that the iteration was proceeding in the proper direction. If not, the computer was stopped, one card in the program changed, and the process again started. The card change then assured the proper trend for the iteration.

The next chapter shows an example equation mechanized by this method.

CHAPTER VI

MECHANIZATION OF AN EXAMPLE EQUATION FOR FIVE POSITIONS BY THE COMPUTER ITERATION METHOD

The equation used for this illustration was the same one used in Trial 9. That is:

$$f(x) = 3x^2 + 5x \qquad 5 \leq x \leq 10$$

The first assumed ranges were:

$$\alpha_{15} = 60.0$$
$$\beta_{15} = 30.0$$
$$\sigma_{15} = 80.0$$

However, it was observed that the resulting σ_{15} was below the 80.0 degrees desired and decreasing as β_{15} was decreased. Therefore, a higher value of $\beta_{15} = 53.0$ degrees was assumed as a starting point for the iteration. (This value could have been any value greater than 45.748 degrees, which is the value previously found by the iteration to satisfy all closure equations. Of course, the resultant β_{15} value is not known normally. In such cases, assumed values would be increased until the resultant σ_{15} becomes greater than 80.0 degrees.) With assumed values given below, the iteration made four loops to completion.

Assumed Ranges:

$$\alpha_{15} = 60.0$$
$$\beta_{15} = 53.0$$
$$\sigma_{15} = 80.0$$

Functions: $\alpha_{12} = \frac{1}{4}\alpha_{15}$

$\alpha_{13} = \frac{2}{4}\alpha_{15}$

$\alpha_{14} = \frac{3}{4}\alpha_{15}$

Functions: $\beta_{12} = \frac{1}{4}\beta_{15}$

$\beta_{13} = \frac{2}{4}\beta_{15}$

$\beta_{14} = \frac{3}{4}\beta_{15}$

Position	x	f(x)	α	β	σ
1	5.00	100.0	0	0	0
2	6.25	148.43750	15.0	13.250	15.50
3	7.50	206.250	30.0	26.50	34.0
4	8.75	273.43750	45.0	39.750	55.50
5	10.0	350.0	60.0	53.0	80.0

The following is a list of σ_{15} values and corresponding β_{15}, as the iteration proceeds:

Run No.	Resultant σ_{15}	Corresponding β_{15}	
1	80.120748	53.0	1st Loop
2	80.053207	49.0	β_{15} Decreased By
3	79.988819	45.0	Steps of 4.0
4	80.053207	49.0	
5	80.046845	48.60	
6	80.044037	48.20	
7	80.034246	47.80	
8	80.026618	47.40	2nd Loop
9	80.022906	47.0	β_{15} Decreased By
10	80.012544	46.60	Steps of 0.40
11	80.010898	46.20	
12	80.005307	45.80	
13	79.995360	45.40	
14	80.005307	45.80	
15	80.000572	45.760	3rd Loop
16	79.999239	45.720	β_{15} Decreased By
17	80.000572	45.760	Steps of 0.040
18	80.000245	45.7560	4th Loop
19	80.000231	45.7520	β_{15} Decreased By
20	79.999764	45.7480	Steps of 0.0040

Thus, the iteration made four loops with the value of the resulting σ_{15} converging upon the desired value of 80.0 degrees. At the end of the fourth loop the resulting mechanism was given by:

Results:

$$Z_2 = -4.0454520 - i0.13609441$$

$$Z_3 = 4.3312001 - i0.48042940$$

$$Z_4 = 0.71514450 + i0.61720100$$

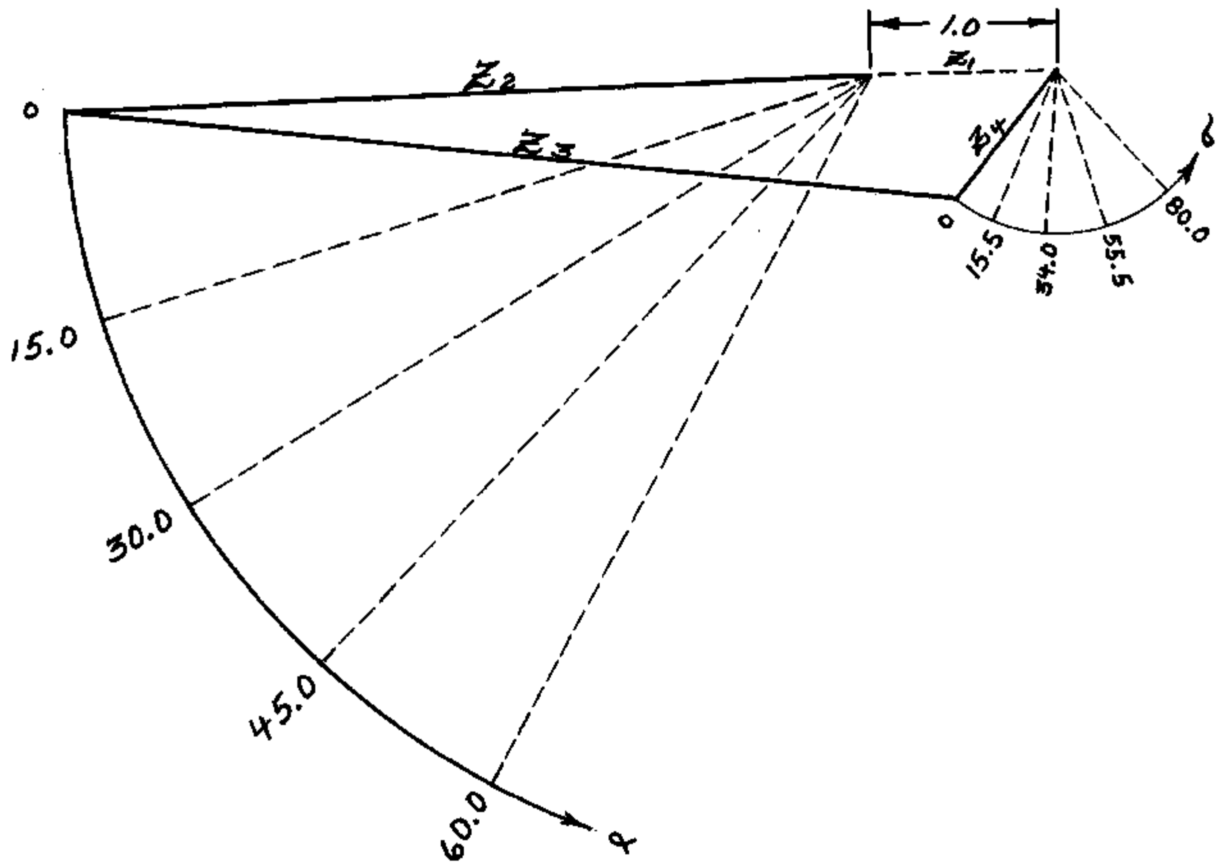

Figure 4. Mechanism Satisfying Given Function For Five Positions.

This mechanism satisfied five positions for the given function.. But if for some reason this mechanism was not satisfactory because of space considerations, etc., a different result could have been obtained by changing either the assumed range of crank (α_{15}) or of follower (σ_{15}). As in the above mechanism, a large crank and coupler as compared with the follower might have been undesirable, or possibly a larger spread of the range of the follower would have been desired. As a second part of this example, another mechanism was designed for the same function, only assuming a different value for the range of the crank (α_{15}). Of course, a different β_{15} would result, so to save time a new starting value for β_{15} was assumed.

Assumed Ranges:

$$\alpha_{15} = 120.0$$
$$\beta_{15} = 100.0$$
$$\sigma_{15} = 80.0$$

Functions: $\alpha_{12} = \frac{1}{4}\,\alpha_{15}$

$\alpha_{13} = \frac{2}{4}\,\alpha_{15}$

$\alpha_{14} = \frac{3}{4}\,\alpha_{15}$

Functions: $\beta_{12} = \frac{1}{4}\,\beta_{15}$

$\beta_{13} = \frac{2}{4}\,\beta_{15}$

$\beta_{14} = \frac{3}{4}\,\beta_{15}$

Mechanize the function:

$$f(x) = 3x^2 + 5x \qquad 5 \leq x \leq 10$$

Position	x	f(x)	α	β	σ
1	5.00	100.0	0	0	0
2	6.25	148.43750	30.0	25.0	15.50
3	7.50	206.250	60.0	50.0	34.0
4	8.75	273.43750	90.0	75.0	55.50
5	10.0	350.0	120.0	100.0	80.0

The following is a list of resultant values of σ_{15} with corresponding values of β_{15}, as the iteration was performed:

Run No.	Resultant σ_{15}	Corresponding β_{15}	
1	80.161999	100.0	1st Loop: β_{15} Decreased By Steps of 4.0
2	80.042084	96.0	
3	79.920222	92.0	
4	80.042084	96.0	
5	80.029726	95.60	2nd Loop: β_{15} Decreased By Steps of 0.40
6	80.017545	95.20	
7	80.005243	94.80	
8	79.993629	94.40	
9	80.005243	94.80	

Run No.	Resultant σ_{15}	Corresponding β_{15}	
10	80.004418	94.760	
11	80.003324	94.720	3^{rd} Loop
12	80.001823	94.680	β_{15} Decreased By
13	80.000430	94.640	Steps of 0.040
14	79.999708	94.600	
15	80.000430	94.640	
16	80.000167	94.6360	
17	80.000131	94.6320	
18	80.000120	94.6280	4^{th} Loop
19	80.000081	94.6240	β_{15} Decreased By
20	80.000072	94.6200	Steps of 0.0040
21	80.000047	94.6160	
22	80.000001	94.6120	
23	79.999977	94.6080	

The resulting value of $\beta_{15} = 94.612$ degrees corresponds to the resulting value of $\sigma_{15} = 80.000001$ degrees. The mechanism for this was given by:

Results:

$$Z_2 = 0.35827410 + i\ 3.8790479$$

$$Z_3 = -2.2692823 - i\ 6.5969343$$

$$Z_4 = 2.9107710 + i\ 2.7257753$$

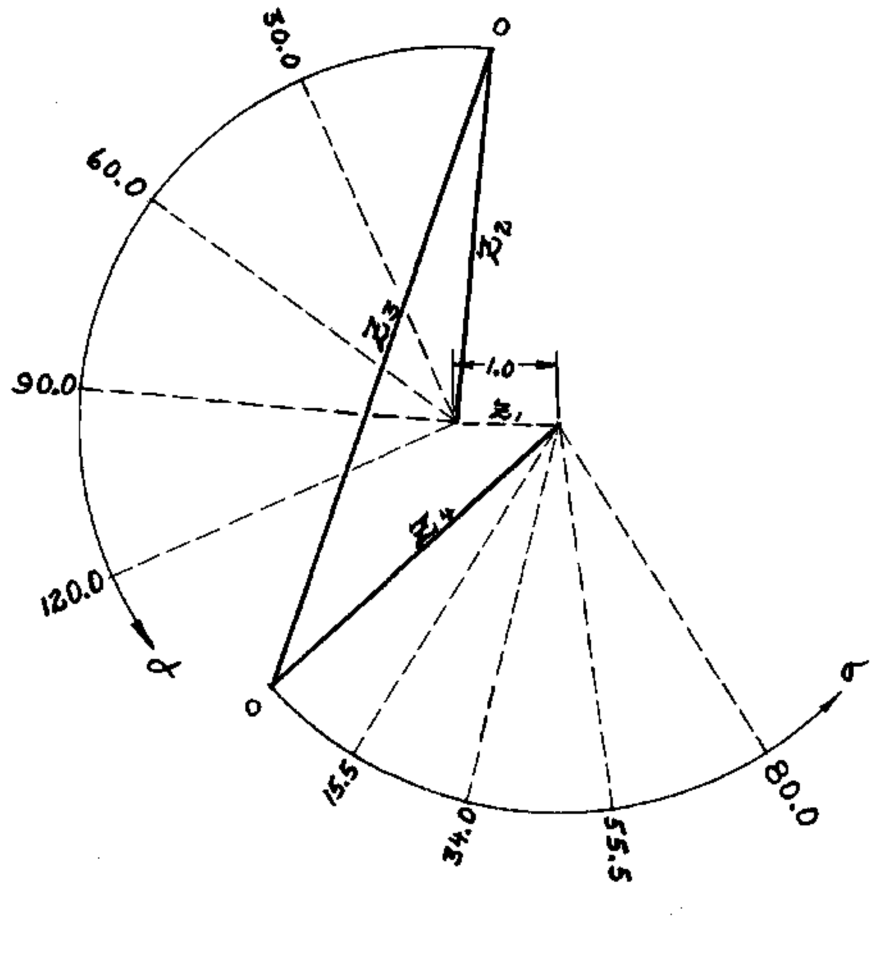

Figure 5. Mechanism Satisfying Given Function For Five Positions.

The resulting mechanism was entirely different from the one previously designed, which also mechanized the given function for five positions. Link Z_1 in both of these mechanisms was equal to 1.0 + i 0. This yielded a simple check that solutions of equations 19, 20, and 21 were correct results for Z_2, Z_3, and Z_4. As a check that the iteration gave a valid solution for all four closure equations, 19 through 22, it was shown that the resulting value for the range and all corresponding values for the equal sub-ranges of the coupler caused the following fourth order determinant to equal zero. The determinant was written for closure equations 19 through 22, repeated below with $Z_1 = 1.0$ for ease of inspection.

$$R\alpha_{12}Z_2 + R\beta_{12}Z_3 + R\sigma_{12}Z_4 = 1.0 \qquad \text{(Eqn 19)}$$

$$R\alpha_{13}Z_2 + R\beta_{13}Z_3 + R\sigma_{13}Z_4 = 1.0 \qquad \text{(Eqn 20)}$$

$$R\alpha_{14}Z_2 + R\beta_{14}Z_3 + R\sigma_{14}Z_4 = 1.0 \qquad \text{(Eqn 21)}$$

$$R\alpha_{15}Z_2 + R\beta_{15}Z_3 + R\sigma_{15}Z_4 = 1.0 \qquad \text{(Eqn 22)}$$

The fourth order determinant is:

$$\begin{vmatrix} R_{\alpha_{12}} & R_{\beta_{12}} & R_{\sigma_{12}} & -1.0 \\ R_{\alpha_{13}} & R_{\beta_{13}} & R_{\sigma_{13}} & -1.0 \\ R_{\alpha_{14}} & R_{\beta_{14}} & R_{\sigma_{14}} & -1.0 \\ R_{\alpha_{15}} & R_{\beta_{15}} & R_{\sigma_{15}} & -1.0 \end{vmatrix}$$

By substituting values for the rotation in the last example, as found on page 40, the determinant becomes:

$$\begin{vmatrix} R_{30} & R_{23.653} & R_{15.5} & -1.0 \\ R_{60} & R_{47.306} & R_{34.0} & -1.0 \\ R_{90} & R_{70.959} & R_{55.5} & -1.0 \\ R_{120} & R_{94.612} & R_{80.0} & -1.0 \end{vmatrix}$$

Note that the value of 94.612 degrees is used for the value of β_{15}, and the sub-ranges are equally divided parts thereof. Expanding the above by the row-column method gives:

$$-1.0 \begin{vmatrix} R_{60} & R_{47.306} & R_{34.0} \\ R_{90} & R_{70.959} & R_{55.5} \\ R_{120} & R_{94.612} & R_{80.0} \end{vmatrix} \quad -1.0 \begin{vmatrix} R_{30} & R_{23.653} & R_{15.5} \\ R_{90} & R_{70.959} & R_{55.5} \\ R_{120} & R_{94.612} & R_{80.0} \end{vmatrix}$$

$$-1.0 \begin{vmatrix} R_{30} & R_{23.653} & R_{15.5} \\ R_{60} & R_{47.306} & R_{34.0} \\ R_{120} & R_{94.612} & R_{80.0} \end{vmatrix} \quad -1.0 \begin{vmatrix} R_{30} & R_{23.653} & R_{15.5} \\ R_{60} & R_{47.306} & R_{34.0} \\ R_{90} & R_{70.959} & R_{55.5} \end{vmatrix}$$

These third order determinants were further expanded by substituting the identity $R\phi = \cos\phi + i\sin\phi$ for each of the rotations. The expanded forms are not presented, however, because they are very lengthy. The final numerical solution, of course, involves two parts: an imaginary part and a real part. The solution was indeed very close to zero, and within the accuracy of the significant figures in the computer results for β_{15} (five figures), it is equal to zero.

The result is:

$$0.0000627 - i\ 0.000091 \approx 0$$

Therefore, it was shown that this iteration method did yield valid results for five positions.

CHAPTER VII

OUTLINE ILLUSTRATING THE USE OF IBM 650 COMPUTER FOR FUNCTION GENERATION OF FIVE POSITIONS BY ITERATION

This final chapter is presented as a guide to enable anyone familiar with equation mechanization to use this method without studying all the theory involved in the process. Directions for this method follow:

1. Assume desired ranges for the crank (α_{15}) and follower (σ_{15}), and a starting value for the range of the coupler (β_{15}). Recommended ranges are:

   Follower: 70 to 100 degrees

   Crank: 60 to 120 degrees

   Coupler: Usually about 20% less than the range of the crank

2. Construct a table in the following form:

Position	x	f(x)	Crank Rotation	Coupler Rotation	Follower Rotation
1			0	0	0
2			$\frac{1}{4}\alpha_{15}$	$\frac{1}{4}\beta_{15}$	σ_{12}
3			$\frac{2}{4}\alpha_{15}$	$\frac{2}{4}\beta_{15}$	σ_{13}
4			$\frac{3}{4}\alpha_{15}$	$\frac{3}{4}\beta_{15}$	σ_{14}
5			α_{15}	β_{15}	σ_{15}

The desired full range of x is equally divided between the five positions; then from the equation to be mechanized the corresponding values for $f(x)$ are determined. The values for the sub-rotations of the crank and the coupler are equal parts of the assumed values for the ranges of these links. For example: If α_{15} is assumed to be 60.0 degrees, then $\alpha_{12} = 15.0$, $\alpha_{13} = 30.0$, and $\alpha_{14} = 45.0$ degrees. Of course, values for σ_{12}', σ_{13}', and σ_{14}' are <u>proportional</u> parts of the assumed full range of the follower (σ_{15}'), as determined by the equation to be mechanized.

3. With this table complete, cut the data cards for the I.B.M. 650 computer, according to the Bell General Purpose System. The data input cards are shown in Appendix C. These data go on cards 374 and 375 in the following address:

	Address	Data
Card 374	701	$\frac{1}{4}\alpha_{15}$
	702	$\frac{2}{4}\alpha_{15}$
	703	$\frac{3}{4}\alpha_{15}$
	704	$\frac{1}{4}\beta_{15}$
	705	$\frac{2}{4}\beta_{15}$
	706	$\frac{3}{4}\beta_{15}$

	Address	Data	
Card 375	707	σ_{12}	
	708	σ_{13}	
	709	σ_{14}	
	710	σ_{15}	
	711	4.0000000	Always use
	712	180.00000	these values

4. With these data cards punched and the rest of the program cards punched exactly as given in Appendix C, the program is ready to be run.

5. The first result card will be punched in about 20 seconds after the computation is begun, followed by a second card about four seconds later. This first card gives the value of the full range of the coupler (β_{15}) used in that iterative trial, and the second card gives the resultant value of the follower rotation (σ_{15}) corresponding to the value of β_{15}. Since the computer automatically makes another trial run with reduced values for the coupler ranges, the third card punched is the new value for β_{15}. The fourth card is the second resultant value of the follower rotation (σ_{15}) corresponding to the new β_{15}.

Thus, as the iteration proceeds the odd number resultant cards are the values for the β_{15}'s tried and have an address of 641, and the even cards are the corresponding resultant σ_{15}'s and have an address of 677. The operator must observe at least the first two resultant values of the follower rotation (σ_{15}) and compare them with the desired value of the follower range. If the resultant values are converging, the iteration should be allowed to continue and no further attention is necessary. If the resultant values are diverging from the desired value, then the iteration must be stopped and card 334 A, (see Appendix C, page 65), substituted for card 334 in the program. After this observation, and substitution of 334 A, if necessary, has been made, the iteration process will proceed automatically until the final mechanism is punched, and the computer program stops. The mechanism is punched on three cards as follows:

	Address	Address
Crank:	555	i 556
Coupler:	603	i 604
Follower:	637	i 638

These results represent the components of the links in the conventional complex number notation. As a check on the results other than the rotational one, the ground link resulting when the links of the mechanism are constructed should be equal to 1.0 + i 0.0. The running time for the entire iteration process is usually about 10 to 15 minutes.

A P P E N D I X A

BGPS PROGRAM ORDERS FOR FOUR POSITION CASE

01	+9	800	001	000	P.P. No. 1	
02	+0	353	601	500	i sin α_{12}	
03	+0	354	601	501	cos α_{12}	$\lambda\alpha_{12}$
04	+1	501	607	502	(cos α_{12} - 1)	
05	+0	353	602	503	i sin α_{13}	
06	+0	354	602	504	cos α_{13}	$\lambda\alpha_{13}$
07	+1	504	607	505	(cos α_{13} - 1)	
08	+0	353	603	506	i sin β_{12}	
09	+0	354	603	507	cos β_{12}	$\lambda\beta_{12}$
10	+1	507	607	508	(cos β_{12} - 1)	
11	+0	353	604	509	i sin β_{13}	
12	+0	354	604	510	cos β_{13}	$\lambda\beta_{13}$
13	+1	510	607	511	(cos β_{13} - 1)	
14	+0	353	605	512	i sin σ'_{12}	
15	+0	354	605	513	cos σ'_{12}	$\lambda\sigma'_{12}$
16	+1	513	607	514	(cos σ'_{12} - 1)	
17	+0	353	606	515	i sin σ'_{13}	
18	+0	354	606	516	cos σ'_{13}	$\lambda\sigma'_{13}$
19	+1	516	607	517	(cos σ'_{13} - 1)	
20	-2	500	515	518	-(sin α_{12})(sin σ'_{13})	
21	+2	500	517	519	(i sin α_{12})(cos σ'_{13} - 1)	
22	+2	502	515	520	(i sin σ'_{13})(cos α_{12} - 1)	
23	+2	502	517	521	(cos α_{12} - 1)(cos σ'_{13} - 1)	
24	+1	518	521	522	518 + 521	

BGPS PROGRAM ORDERS FOR FOUR POSITION CASE (Continued)

25	+1	519	520	523	519 + 520	
26	+2	503	512	524	$-(\sin \alpha_{13})(i \sin \sigma_{12})$	
27	-2	503	514	525	$-(i \sin \alpha_{13})(\cos \sigma_{12} - 1)$	
28	-2	505	512	526	$-(i \sin \sigma_{12})(\cos \sigma_{12} - 1)$	
29	-2	505	514	527	$-(\cos \alpha_{13} - 1)(\cos \sigma_{12} - 1)$	
30	+1	524	527	528	524 + 527	
31	+1	525	526	529	525 + 526	
32	+1	522	528	530	522 + 528 = D	
33	+1	523	529	531	523 + 529 = id	
34	-2	506	515	532	$(i \sin \beta_{12})(i \sin \sigma_{13})$	
35	+2	506	517	533	$(i \sin \beta_{12})(\cos \sigma_{13} - 1)$	
36	+2	508	515	534	$(i \sin \sigma_{13})(\cos \beta_{12} - 1)$	
37	+2	508	517	535	$(\cos \beta_{12} - 1)(\cos \beta_{13} - 1)$	
38	+1	532	535	536	532 + 535	
39	+1	533	534	537	533 + 534	
40	+2	509	512	538	$-(i \sin \beta_{13})(i \sin \sigma_{12})$	
41	-2	509	514	539	$-(i \sin \beta_{13})(\cos \beta_{12} - 1)$	
42	-2	511	512	540	$-(i \sin \sigma_{12})(\cos \beta_{13} - 1)$	
43	-2	511	514	541	$-(\cos \beta_{13} - 1)(\cos \sigma_{12} - 1)$	
44	+1	538	541	542	538 + 542	
45	+1	539	540	543	539 + 540	
46	+1	536	542	544	536 + 542 = (n)	NUM.
47	+1	537	543	545	537 + 543 = i(N)	Z_2
48	+2	530	544	546	D(n)	

BGPS PROGRAM ORDERS FOR FOUR POSITION CASE (Continued)

49	+2	530	545	547	(D)i(N)
50	-2	531	544	548	-(id)(n)
51	+2	531	545	549	(id)(iN)
52	+1	546	549	550	$(\ n\ )Z_2$
53	+1	547	548	551	$(iN)Z_2$
54	+2	530	530	552	D^2
55	+2	531	531	555	$(id)^2$
56	+1	552	553	554	552 + 553
57	+3	550	554	555	N_2
58	+3	551	554	556	$i\eta_2$
58a	+7	300	555	556	Punch Z_2
59	-2	500	509	557	$-(i\ \sin\alpha_{12})(i\ \sin\beta_{13})$
60	+2	500	511	558	$(i\ \sin\alpha_{12})(\cos\beta_{13} - 1)$
61	+2	502	509	559	$(i\ \sin\beta_{13})(\cos\alpha_{12} - 1)$
62	+2	502	511	560	$(\cos\alpha_{12} - 1)(\cos\beta_{13} - 1)$
63	+1	557	560	561	557 + 560
64	+1	558	559	562	558 + 559
65	+2	503	506	563	$-(i\ \sin\alpha_{13})(i\ \sin\beta_{12})$
66	-2	503	508	564	$-(i\ \sin\alpha_{13})(\cos\beta_{12} - 1)$
67	-2	505	506	565	$-(i\ \sin\beta_{12})(\cos\alpha_{13} - 1)$
68	-2	505	508	566	$-(\cos\alpha_{13} - 1)(\cos\beta_{12} - 1)$
69	+1	563	566	567	563 + 566
70	+1	564	565	568	564 + 565

BGPS PROGRAM ORDERS FOR FOUR POSITION CASE (Concluded)

71	+1	561	567	569	561 + 567 = (n)	NUM. Z_4
72	+1	562	568	570	562 + 568 = i(N)	
73	+2	530	569	571	D(n)	
74	+2	530	570	572	(D)i(N)	
75	-2	531	569	573	-(id)(n)	
76	+2	531	570	574	-(id)(i(N))	
77	+1	571	574	575	$(n)_{Z_4}$	Numerator Z_4
79	+1	572	573	576	$(i\eta)_{Z_4}$	
80	+3	575	554	577	N_4	Final Z_4
81	+3	576	554	578	$i\eta_4$	
82	+7	300	577	578	Punch Z_4	
83	+0	000	000		Stop	

DATA INPUT

84	601	6	+2	000	000	051	α_{12}
	602		+4	000	000	051	α_{13}
	603		+1	000	000	051	β_{12}
	604		+2	000	000	051	β_{13}
	605		+2	647	134	050	σ_{12}
	606		+2	085	728	451	σ_{13}
85	607	1	-1	000	000	050	-1.0000000
86	000	0					Start

A P P E N D I X B

TRIAL 9

It was observed during the preceding trials, especially in trails 1 and 2, that the coupler's resulting full range of rotation (β_{14}), when the mechanism's crank was rotated its full range, followed a rather definite pattern, depending on the resulting rotation of the follower. Therefore, it was decided to investigate what effect the assumed range (β_{14}) of the coupler had on the resulting range of the follower (σ_{14}). The trial was conducted by varying the assumed value of β_{14} over a wide range, while holding α_{14} and σ_{14} constant. The crank and the follower generated the given equation, and it was desired to cause the resulting value of σ_{14}, (to approach the assumed value for σ_{14}), when the crank was rotated the assumed value α_{14}. Whatever value β_{14} resulted was of little concern, so long as it gave a reasonable mechanism. The sub-ranges of the crank and the coupler were equal divisions of their full assumed ranges.

Variables: β_{14}

Constants: α_{14}, σ_{14}

Functions: $\beta_{12} = \frac{1}{3}\beta_{14}$

$\beta_{13} = \frac{2}{3}\beta_{14}$

Functions: $\alpha_{12} = \frac{1}{3}\alpha_{14}$

$\alpha_{13} = \frac{2}{3}\alpha_{14}$

The mechanized function was:

$$f(x) = 3x^2 + 5x \qquad 5 \leq x \leq 10$$

Assumed Ranges: $\alpha_{14} = 60.0$

$\beta_{14} = 7.5$

$\sigma_{14} = 80.0$

Position	x	f(x)	α	β	σ
1	5.0	100.0	0	0	0
2	6.6666666	166.66666	20.0	2.50	21.333333
3	8.3333333	249.99999	40.0	5.0	47.999999
4	10.0	350.0	60.0	7.50	80.0

Results: $Z_2 = -0.10544029 - i\ 0.15013209$

$Z_3 = 1.0000000 + i\ 0.0$

$Z_4 = 0.0076015077 + i\ 0.15551958$

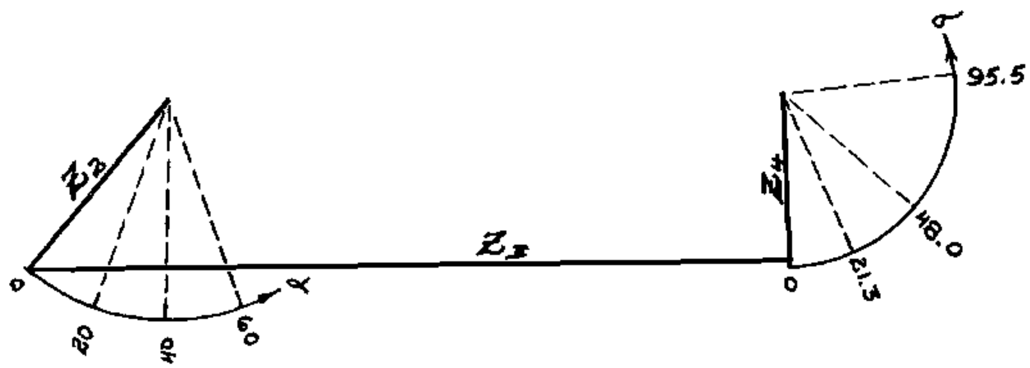

Figure 6. Mechanism Showing Resultant Follower Rotation for β_{14} Equal to 7.50 Degrees

The resulting follower rotation was 95.5 degrees as shown. Next, the assumed full range of the coupler (β_{14}) was increased to 12.0 degrees, all else remaining the same.

Assumed Ranges: $\alpha_{14} = 60.0$

$\beta_{14} = 12.0$

$\sigma_{14} = 80.0$

Position	x	f(x)	α	β	σ
1	5.0	100.0	0	0	0
2	6.6666666	166.66666	20.0	4.0	21.333333
3	8.3333333	249.99999	40.0	8.0	47.999999
4	10.0	350.0	60.0	12.0	80.0

Results: $Z_2 = -0.32552396 - i\,0.14815410$

$Z_3 = 1.0000000 + i\,0.0$

$Z_4 = 0.12059295 + i\,0.16384068$

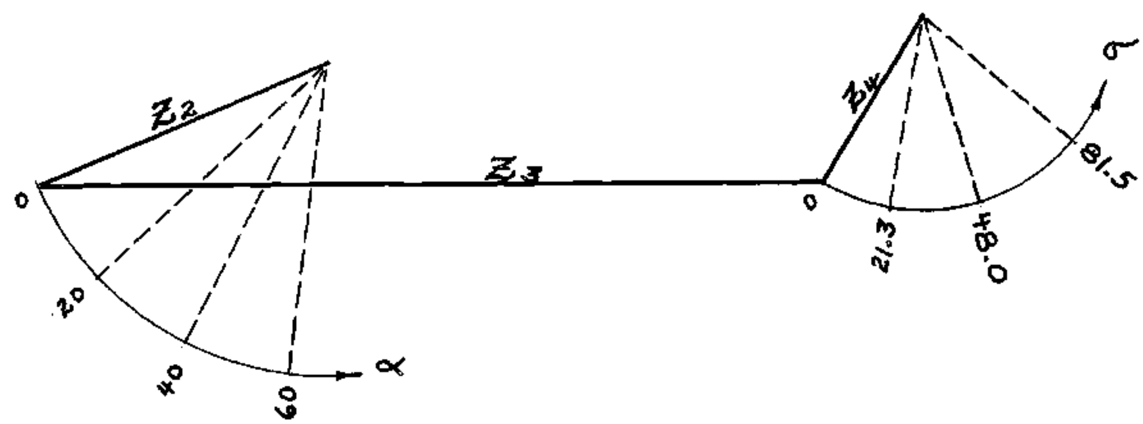

Figure 7. Mechanism Showing Resultant Follower Rotation For β_{14} Equal to 12.0 Degrees

For this mechanism the resulting σ_{14} was much closer to the desired σ_{14} of 80.0 degrees, so this trend of increasing β_{14} was continued, with the new assumed value for $\beta_{14} = 21.0$ degrees.

Assumed Ranges: $\alpha_{14} = 60.0$

$\beta_{14} = 21.0$

$\sigma_{14} = 80.0$

Position	x	f(x)	α	β	σ
1	5.0	100.0	0	0	0
2	6.6666666	166.66666	20.0	7.0	21.333333
3	8.3333333	249.99999	40.0	14.0	47.999999
4	10.0	350.0	60.0	21.0	80.0

Results: Z_2 = −0.51593345 − i 0.21987505

Z_3 = 1.0000000 + i 0.0

Z_4 = 0.15914800 + i 0.24177916

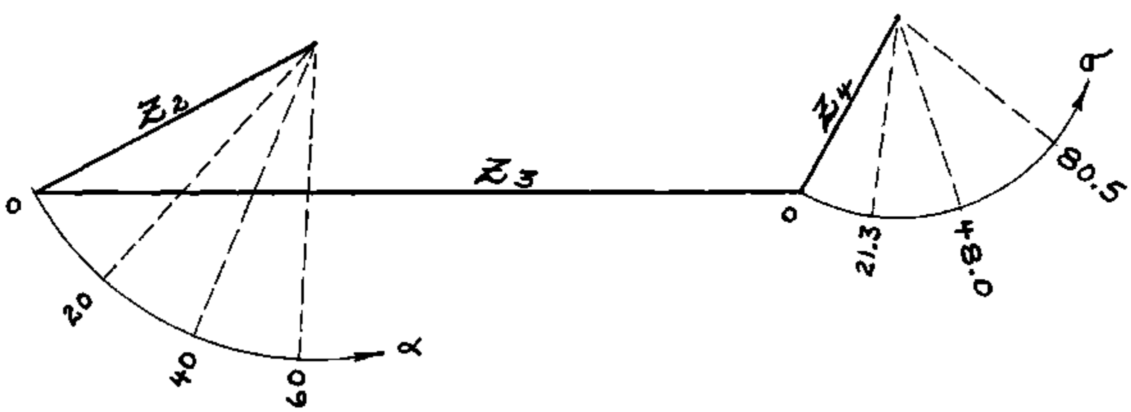

Figure 8. Mechanism Showing Resultant Follower Rotation For β_{14} Equals to 21.0 Degrees

The resulting σ_{14} was now 80.5 degrees, when the crank was rotated the assumed range for α_{14} of 60.0 degrees. This resulting σ_{14} was still approaching the desired σ_{14} of 80.0 degrees, so the method was continued. The value of β_{14} was next assumed to be 30.0 degrees.

Assumed Ranges: $\alpha_{14} = 60.0$

$\beta_{14} = 30.0$

$\sigma_{14} = 80.0$

Position	x	f(x)	α	β	σ
1	5.0	100.0	0	0	0
2	6.6666666	166.66666	20.0	10.0	21.333333
3	8.3333333	249.99999	40.0	20.0	47.999999
4	10.0	350.0	60.0	30.0	80.0

Results:

$Z_2 = -0.66796212 - i\ 0.25087035$

$Z_3 = 1.0000000 + i\ 0.0$

$Z_4 = 0.16077999 + i\ 0.27454311$

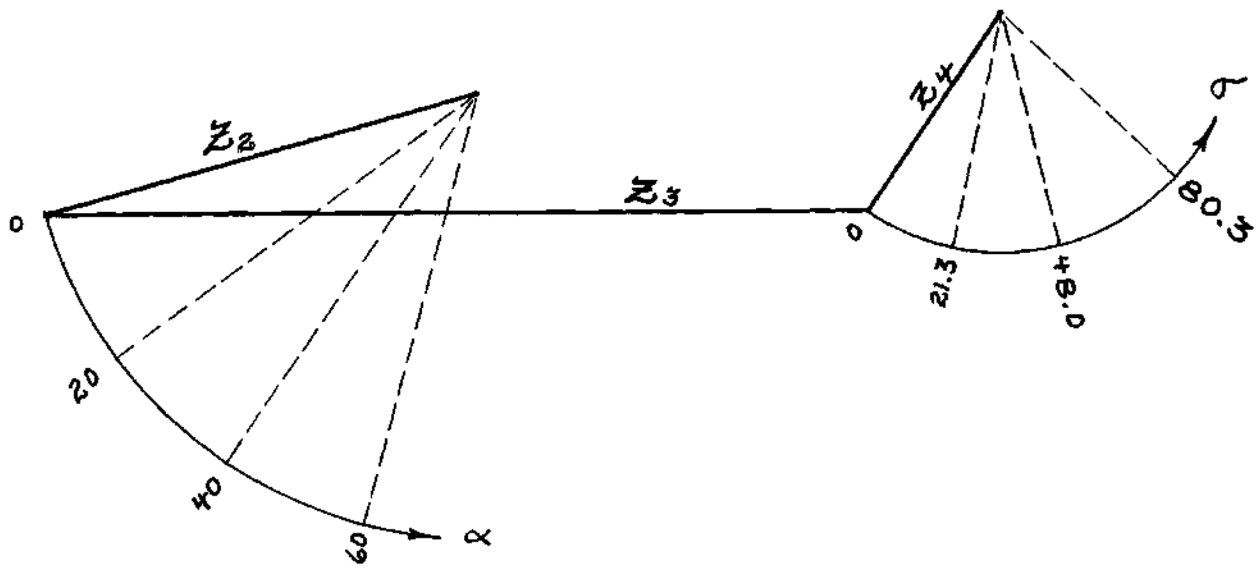

Figure 9. Mechanism Showing Resultant Follower Rotation For β_{14} Equal to 30.0 Degrees

The resultant σ_{14} of 80.3 degrees was now very close to the desired σ_{14} of 80.0 degrees. Therefore, the assumed value of the range of the coupler was further increased to β_{14} of 36.0 degrees.

Assumed Ranges: $\alpha_{14} = 60.0$

$\beta_{14} = 36.0$

$\sigma_{14} = 80.0$

Position	x	f(x)	α	β	σ
1	5.0	100.0	0	0	0
2	6.6666666	166.66666	20.0	12.0	21.333333
3	8.3333333	249.99999	40.0	24.0	47.999999
4	10.0	350.0	60.0	36.0	80.0

Results:

$$Z_2 = -0.75174948 - i\ 0.24626717$$
$$Z_3 = 1.0000000 + i\ 0.0$$
$$Z_4 = 0.14504296 + i\ 0.26876128$$

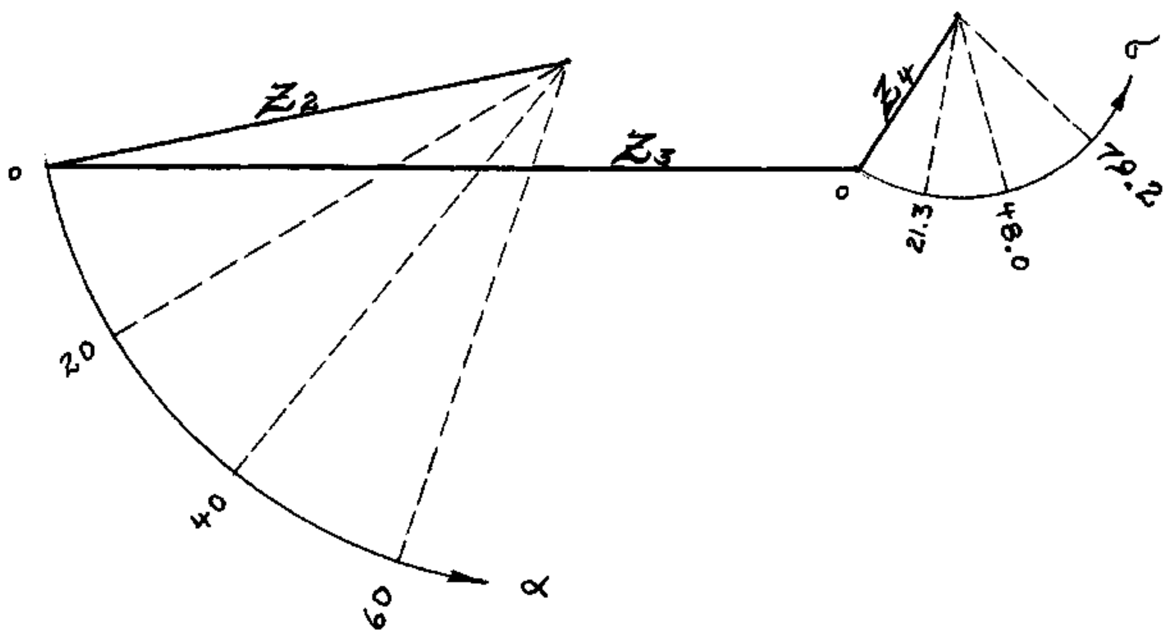

Figure 10. Mechanism Showing Resultant Follower Rotation For β_{14} Equal to 36.0 Degrees

The resulting σ_{14} of 79.2 degrees was the one which had been sought, since it shows that the resulting value of σ_{14} has "crossed over" and is now less than the desired 80.0 degrees.

Conclusion.--Between 30.0 and 36.0 degrees there lies a value for β_{14} which will cause the resulting σ_{14} to very closely approximate the desired value of σ_{14} equal to 80.0 degrees. This value, when found, along with Z_2 and Z_4, would satisfy all three closure equations 11, 12, and 13, and the third order matrix as developed in general terms on page 9 would go to zero. In this trial an increase of the assumed range of β_{14} caused the resulting σ_{14} to decrease; however, it was found in other cases tried that an increase in the assumed values for β_{14} did not always cause a decrease in the resulting σ_{14}. Whether the resulting σ_{14} increased or decreased with an increase in β_{14} depended on the equation being mechanized. The values of β_{14} were either increased or decreased, depending on the particular equation, in such a manner as to cause the resulting σ_{14} to approach the desired σ_{14}.

A P P E N D I X   C

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE

01	+9	800	001	000	P.P. No. 1	
02	+0	354	701	400	$\cos\alpha_{12}$	$R\alpha_{12}$
03	+0	353	701	401	$i\sin\alpha_{12}$	
04	+0	354	702	402	$\cos\alpha_{13}$	$R\alpha_{13}$
05	+0	353	702	403	$i\sin\alpha_{13}$	
06	+0	354	703	404	$\cos\alpha_{14}$	$R\alpha_{14}$
07	+0	353	703	405	$i\sin\alpha_{14}$	
08	+0	354	704	406	$\cos\beta_{12}$	$R\beta_{12}$
09	+0	353	704	407	$i\sin\beta_{12}$	
10	+0	354	705	408	$\cos\beta_{13}$	$R\beta_{13}$
11	+0	353	705	409	$i\sin\beta_{13}$	
12	+0	354	706	410	$\cos\beta_{14}$	$R\beta_{14}$
13	+0	353	706	411	$i\sin\beta_{14}$	
14	+0	354	707	412	$\cos\sigma_{12}$	$R\sigma_{12}$
15	+0	353	707	413	$i\sin\sigma_{12}$	
16	+0	354	708	414	$\cos\sigma_{13}$	$R\sigma_{13}$
17	+0	353	708	415	$i\sin\sigma_{13}$	
18	+0	354	709	416	$\cos\sigma_{14}$	$R\sigma_{14}$
19	+0	353	709	417	$i\sin\sigma_{14}$	
20	+2	400	408	418	$(\cos\alpha_{12})\times(\cos\beta_{13}) = 418$	
21	+2	400	409	419	$(\cos\alpha_{12})\times(i\sin\beta_{13}) = 419$	
22	+2	401	408	420	$(i\sin\alpha_{12})\times(\cos\beta_{13}) = 420$	

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

23	-2	401	409	421	$-(i \sin\alpha_{12}) x (i \sin\beta_{12}) = 421$
24	+1	419	420	422	$419 + 420 = 422$
25	+1	418	421	423	$418 + 421 = 423$
26	+2	423	416	424	$423 \text{ x } (\cos \sigma_{14}) = 424$
27	+2	423	417	425	$423 \text{ x } (i \sin\sigma_{14}) = 425$
28	+2	422	416	426	$422 \text{ x } (\cos\sigma_{14}) = 426$
29	-2	422	417	427	$-422 \text{ x } (i \sin\sigma_{13}) = 427$
30	+1	425	426	428	$425 + 426 = 428$
31	+1	424	427	429	$424 + 427 = 429$
32	+2	402	410	430	$(\cos\alpha_{13}) x (\cos\beta_{14}) = 430$
33	+2	402	411	431	$(\cos\alpha_{13}) x (i \sin\beta_{14}) = 431$
34	+2	403	410	432	$(i \sin\alpha_{13}) x (\cos\beta_{14}) = 432$
35	-2	403	411	433	$-(i \sin\alpha_{13}) x (i \sin\beta_{14}) = 433$
36	+1	431	432	434	$431 + 432 = 434$
37	+1	430	433	435	$430 + 433 = 435$
38	+2	435	412	436	$435 \text{ x } (\cos\sigma_{12}) = 436$
39	+2	435	413	437	$435 \text{ x } (i \sin\sigma_{12}) = 437$
40	+2	434	412	438	$434 \text{ x } (\cos \sigma_{12}) = 438$
41	-2	434	413	439	$-434 \text{ x } (i \sin\sigma_{12}) = 439$
42	+1	437	438	440	$437 + 438 = 440$
43	+1	436	439	441	$436 + 439 = 441$
44	+2	406	414	442	$(\cos\beta_{12}) x (\cos \sigma_{13}) = 442$
45	+2	406	415	443	$(\cos\beta_{12}) x (i \sin\sigma_{13}) = 443$
46	+2	407	414	444	$(i \sin\beta_{12}) x (\cos\sigma_{13}) = 444$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

47	-2	407	415	445	$-(i \sin\beta_{12}) x (i \sin\sigma_{13}) = 445$
48	+1	443	444	446	443 + 444 = 446
49	+1	442	445	447	442 + 445 = 447
50	+2	447	404	448	$447 \times (\cos\alpha_{14}) = 448$
51	+2	447	405	449	$447 \times (i \sin\alpha_{14}) = 449$
52	+2	446	404	450	$446 \times (\cos\alpha_{14}) = 450$
53	-2	446	405	451	$446 \times (i \sin\alpha_{14}) = 451$
54	+1	449	450	452	449 + 450 = 452
55	+1	448	451	453	448 + 451 = 453
56	+2	412	408	454	$(\cos\sigma_{12}) x (\cos\beta_{13}) = 454$
57	+2	412	409	455	$(\cos\sigma_{12}) x (i \sin\beta_{13}) = 455$
58	+2	413	408	456	$(i \sin\sigma_{12}) x (\cos\beta_{13}) = 456$
59	-2	413	409	457	$-(i \sin\sigma_{12}) x (i \sin\beta_{13}) = 457$
60	+1	455	456	458	455 + 456 = 458
61	+1	454	457	459	454 + 457 = 459
62	+2	459	404	460	$459 \times (\cos\alpha_{14}) = 460$
63	+2	459	405	461	$459 \times (i \sin\alpha_{14}) = 461$
64	+2	458	404	462	$458 \times (\cos\alpha_{14}) = 462$
65	-2	458	405	463	$-458 \times (i \sin\alpha_{14}) = 463$
66	+1	461	462	464	461 + 462 = 464
67	+1	460	463	465	460 + 463 = 465
68	-2	901	464	464	Change sign 464
69	-2	901	465	465	Change sign 465
70	+2	406	402	466	$(\cos\beta_{12}) x (\cos\alpha_{13}) = 466$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

71	+2	406	403	467	$(\cos\beta_{12}) \times (i \sin\alpha_{13}) = 467$
72	+2	407	402	468	$(i \sin\beta_{12}) \times (\cos\alpha_{13}) = 468$
73	-2	407	403	469	$-(i \sin\beta_{12}) \times (i \sin\alpha_{13}) = 469$
74	+1	467	468	470	467 + 468 = 470
75	+1	466	469	471	466 + 469 = 471
76	+2	471	416	472	$471 \times (\cos\sigma_{14}) = 472$
77	+2	471	417	473	$471 \times (i \sin\sigma_{14}) = 473$
78	+2	470	416	474	$470 \times (\cos\sigma_{14}) = 474$
79	-2	470	417	475	$-470 \times (i \sin\sigma_{14}) = 475$
80	+1	473	474	476	473 + 474 = 476
81	+1	472	475	477	472 + 475 = 477
82	-2	901	476	476	Change sign at 476
83	-2	901	477	477	Change sign at 477
84	+2	400	410	478	$(\cos\alpha_{12}) \times (\cos\beta_{14}) = 478$
85	+2	400	411	479	$(\cos\alpha_{12}) \times (i \sin\beta_{14}) = 479$
86	+2	401	410	480	$(i \sin\alpha_{12}) \times (\cos\beta_{14}) = 480$
87	-2	401	411	481	$-(i \sin\alpha_{12}) \times (i \sin\beta_{14}) = 481$
88	+1	479	480	482	479 + 480 = 482
89	+1	478	481	483	478 + 481 = 483
90	+2	483	414	484	$483 \times (\cos\sigma_{13}) = 484$
91	+2	483	415	485	$483 \times (i \sin\sigma_{13}) = 485$
92	+2	482	414	486	$482 \times (\cos\sigma_{13}) = 486$
93	-2	482	415	487	$-482 \times (i \sin\sigma_{13}) = 487$
94	+1	485	486	488	485 + 486 = 488

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

95	+1	484	487	489	484 + 487 = 489
96	-2	901	488	488	Change sign 488
97	-2	901	489	489	Change sign 489
98	+1	429	441	490	429 + 441 = 490
99	+1	490	453	491	490 + 453 = 491
100	+1	465	477	492	465 + 477 = 492
101	+1	492	489	493	492 + 489 = 493
102	+1	491	493	494	N denominator
103	+1	428	440	495	428 + 440 = 495
104	+1	495	452	496	495 + 452 + 496
105	+1	464	476	497	464 + 476 = 497
106	+1	497	488	498	497 + 488 = 498
107	+1	496	498	499	$i\eta$ denominator
108	+2	408	416	500	$(\cos\beta_{13})x(\cos\sigma_{14}) = 500$
109	+2	408	417	501	$(\cos\beta_{13})x(i\ \sin\sigma_{14}) = 501$
110	+2	409	416	502	$(i\ \sin\beta_{13})x(\cos\sigma_{14}) = 502$
111	-2	409	417	503	$-(i\ \sin\beta_{13})x(i\ \sin\sigma_{14}) = 503$
112	+1	501	502	504	501 + 502 = 504
113	+1	500	503	505	500 + 503 = 505
114	+2	406	414	506	$(\cos\beta_{12})x(\cos\sigma_{13}) = 506$
115	+2	406	415	507	$(\cos\beta_{12})x(i\ \sin\sigma_{13}) = 507$
116	+2	407	414	508	$(i\ \sin\beta_{12})x(\cos\sigma_{13}) = 508$
117	-2	407	415	509	$-(i\ \sin\beta_{12})x(i\ \sin\sigma_{13}) = 509$
118	+1	507	508	510	507 + 508 = 510

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

119	+1	506	509	511	506 + 509 = 511
120	+2	410	412	512	$(\cos\beta_{14})x(\cos\sigma_{12}) = 512$
121	+2	410	413	513	$(\cos\beta_{14})x(i \sin\sigma_{12}) = 513$
122	+2	411	412	514	$(i \sin\beta_{14})x(\cos\sigma_{12}) = 514$
123	-2	411	413	515	$-(i \sin\beta_{14})x(i \sin\sigma_{12} = 515$
124	+1	513	514	516	513 + 514 = 516
125	+1	512	515	517	512 + 515 = 517
126	+2	412	408	518	$(\cos\sigma_{12})x(\cos\beta_{13}) = 518$
127	+2	412	409	519	$(\cos\sigma_{12})x(i \sin\beta_{13}) = 519$
128	+2	413	408	520	$(i \sin\sigma_{12})x(\cos\beta_{13}) = 520$
129	-2	413	409	521	$-(i \sin\sigma_{12})x(i \sin\beta_{13}) = 521$
130	+1	519	520	522	519 + 520 = 522
131	+1	518	521	523	518 + 521 = 523
132	-2	901	522	522	Change sign at 522
133	-2	901	523	523	Change sign at 523
134	+2	414	410	524	$(\cos\sigma_{13})x(\cos\beta_{14}) = 524$
135	+2	414	411	525	$(\cos\sigma_{13})x(i \sin\beta_{14}) = 525$
136	+2	415	410	526	$(i \sin\sigma_{13})x(\cos\beta_{14}) = 526$
137	-2	415	411	527	$-(i \sin\sigma_{13})x(i \sin\beta_{14}) = 527$
138	+1	525	526	528	525 + 526 = 528
139	+1	524	527	529	524 + 527 = 529
140	-2	901	528	528	Change sign 528
141	-2	901	529	529	Change sign 529
142	+2	406	416	530	$(\cos\beta_{12})x(\cos\sigma_{14}) = 530$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

143	+2	406	417	531	$(\cos\beta_{12})\times(i\ \sin\sigma_{14}) = 531$
144	+2	407	416	532	$(i\ \sin\beta_{12})\times(\cos\sigma_{14}) = 532$
145	-2	407	417	533	$-(i\ \sin\beta_{12})\times(i\ \sin\sigma_{14}) = 533$
146	+1	531	532	534	531 + 532 = 534
147	+1	530	533	535	530 + 533 = 535
148	-2	901	534	534	Change sign of 534
149	-2	901	535	535	Change sign of 535
150	+1	505	511	536	505 + 511 = 536
151	+1	536	517	537	536 + 517 = 537
152	+1	523	529	538	523 + 529 = 538
153	+1	538	535	539	538 + 535 = 539
154	+1	537	539	540	Numerator N_{Z_2}
155	+1	504	510	541	504 + 510 = 541
156	+1	541	516	542	541 + 516 = 542
157	+1	522	528	543	522 + 528 = 543
158	+1	543	534	544	543 + 534 = 544
159	+1	542	544	545	Numerator $i\eta_{Z_2}$
160	+2	540	494	546	Eliminate (i) from denominator
161	-2	540	499	547	-540 x 499 = 547
162	+2	545	494	548	545 x 494 = 548
163	+2	545	499	549	545 x 499 = 549
164	+1	547	548	550	547 + 548 = 550
165	+1	546	549	551	546 + 549 = 551
166	+2	494	494	552	$(494)^2 = 552$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

167	+2	499	499	553	$(499)^2$ - 553
168	+1	552	553	554	Final value Denominator
169	+3	551	554	555	N_2 Final Z_2
170	+3	550	554	556	$i\eta_2$
172	+2	400	416	557	$(\cos\alpha_{12})x(\cos\sigma_{14})$ - 557
173	+2	400	417	558	$(\cos\alpha_{12})x(i\sin\sigma_{14})$ - 558
174	+2	401	416	559	$(i\sin\alpha_{12})x(\cos\sigma_{14})$ - 559
175	-2	401	417	560	$-(i\sin\alpha_{12})x(i\sin\sigma_{14})$ - 560
176	+1	558	559	561	558 + 559 - 561
177	+1	557	560	562	557 + 560 - 562
178	+2	414	404	563	$(\cos\sigma_{13})x(\cos\alpha_{14})$ - 563
179	+2	414	405	564	$(\cos\sigma_{13})x(i\sin\alpha_{14})$ - 564
180	+2	415	404	565	$(i\sin\sigma_{13})x(\cos\alpha_{14})$ - 565
181	-2	415	405	566	$-(i\sin\sigma_{13})x(i\sin\alpha_{14})$ - 566
182	+1	564	565	567	564 + 565 - 567
183	+1	563	566	568	563 + 566 - 568
184	+2	402	412	569	$(\cos\alpha_{13})x(\cos\sigma_{12})$ - 569
185	+2	402	413	570	$(\cos\alpha_{13})x(i\sin\sigma_{12})$ - 570
186	+2	403	412	571	$(i\sin\alpha_{13})x(\cos\sigma_{12})$ - 571
187	-2	403	413	572	$(i\sin\alpha_{13})x(i\sin\sigma_{12})$ - 572
188	+1	570	571	573	570 + 571 - 573
189	+1	569	572	574	569 + 572 - 574
190	+2	412	404	575	$(\cos\sigma_{12})x(\cos\alpha_{14})$ - 575
191	+2	412	405	576	$(\cos\sigma_{12})x(i\sin\alpha_{14})$ - 576
192	+2	413	404	577	$(i\sin\sigma_{12})x(\cos\alpha_{14})$ - 577

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

193	-2	413	405	578	$-(i \sin\sigma_{12})x(i \sin\alpha_{14}) = 578$
194	+1	576	577	579	576 + 577 = 579
195	+1	575	578	580	575 + 578 = 580
196	-2	901	579	579	Change sign of 579
197	-2	901	580	580	Change sign of 580
198	+2	414	400	581	$(\cos\sigma_{13})x(\cos\alpha_{12}) = 581$
199	+2	414	401	582	$(\cos\sigma_{13})x(i \sin\alpha_{12}) = 582$
200	+2	415	400	583	$(i \sin\sigma_{13})x(\cos\alpha_{12}) = 583$
201	-2	415	401	584	$-(i \sin\sigma_{13})x(i \sin\alpha_{12}) = 584$
202	+1	582	583	585	582 + 583 = 585
203	+1	581	584	586	581 + 584 = 586
204	-2	901	585	585	Change sign of 585
205	-2	901	586	586	Change sign of 586
206	+2	402	416	587	$(\cos\alpha_{13})x(\cos\sigma_{14}) = 587$
207	+2	402	417	588	$(\cos\alpha_{13})x(i \sin\sigma_{14}) = 588$
208	+2	403	416	589	$(i \sin\alpha_{13})x(\cos\sigma_{14}) = 589$
209	-2	403	417	590	$-(i \sin\alpha_{13})x(i \sin\sigma_{14}) = 590$
210	+1	588	589	591	588 + 589 = 591
211	+1	587	590	592	587 + 590 = 592
212	-2	901	591	591	Change sign of 591
213	-2	901	592	592	Change sign of 592
214	+1	562	568	593	562 + 568 = 593
215	+1	593	574	593	593 + 574 = 593
216	+1	580	586	594	580 + 586 = 594
217	+1	594	592	594	594 + 592 = 594

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

218	+1	593	594	594	593 + 594 = 594
219	+1	561	567	595	561 + 567 = 595
220	+1	595	573	595	595 + 573 = 595
221	+1	579	585	596	579 + 585 = 596
222	+1	596	591	596	596 + 591 = 596
223	+1	595	596	596	595 + 596 = 596
224	+2	594	494	597	594 x 494 = 597
225	-2	594	499	598	-594 x 499 = 598
226	+2	596	494	599	596 x 494 = 599
227	+2	596	499	600	596 x 499 = 600
228	+1	598	599	601	598 + 599 = 601
229	+1	597	600	602	597 + 600 = 602
230	+3	602	554	603	N_2 Z_3
231	+3	601	554	604	$i\eta_3$
233	+2	400	408	605	$(\cos\alpha_{12})$x$(\cos\beta_{13})$ = 605
234	+2	400	409	606	$(\cos\alpha_{12})$x$(i\ \sin\beta_{13})$ = 606
235	+2	401	408	607	$(i\ \sin\alpha_{12})$x$(\cos\beta_{13})$ = 607
236	-2	401	409	608	$-(i\ \sin\alpha_{12})$x$(i\ \sin\beta_{13})$ = 608
237	+1	606	607	607	606 + 607 = 607
238	+1	605	608	608	605 + 608 = 608
239	+2	406	404	609	$(\cos\beta_{12})$x$(\cos\alpha_{14})$ = 609
240	+2	406	405	610	$(\cos\beta_{12})$x$(i\ \sin\alpha_{14})$ = 610
241	+2	407	404	611	$(i\ \sin\beta_{12})$x$(\cos\alpha_{14})$ = 611
242	-2	407	405	612	$-(i\ \sin\beta_{12})$x$(i\ \sin\alpha_{14})$ = 612

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

243	+1	610	611	611	610 + 611 = 611
244	+1	609	612	612	609 + 612 = 612
245	+2	402	410	613	$(\cos\alpha_{13})\times(\cos\beta_{14}) = 613$
246	+2	402	411	614	$(\cos\alpha_{13})\times(i\ \sin\beta_{14}) = 614$
247	+2	403	410	615	$(i\ \sin\alpha_{13})\times(\cos\beta_{14}) = 615$
248	-2	403	411	616	$-(i\ \sin\alpha_{13})\times(i\ \sin\beta_{14}) = 616$
249	+1	614	615	615	614 + 615 = 615
250	+1	613	616	616	613 + 616 = 616
251	+2	408	404	617	$(\cos\beta_{13})\times(\cos\alpha_{14}) = 617$
252	+2	408	405	618	$(\cos\beta_{13})\times(i\ \sin\alpha_{14}) = 618$
253	+2	409	404	619	$(i\ \sin\beta_{13})\times(\cos\alpha_{14}) = 619$
254	-2	409	405	620	$-(i\ \sin\beta_{13})\times(i\ \sin\alpha_{14}) = 620$
255	+1	618	619	619	618 + 619 = 619
256	+1	617	620	620	617 + 620 = 620
257	-2	901	619	619	Change sign of 619
258	-2	901	620	620	Change sign of 620
259	+2	410	400	621	$(\cos\beta_{14})\times(\cos\alpha_{12}) = 621$
260	+2	410	401	622	$(\cos\beta_{14})\times(i\ \sin\alpha_{12}) = 622$
261	+2	411	400	623	$(i\ \sin\beta_{14})\times(\cos\alpha_{12}) = 623$
262	-2	411	401	624	$-(i\ \sin\beta_{14})\times(i\ \sin\alpha_{12}) = 624$
263	+1	622	623	623	622 + 623 = 623
264	+1	621	624	624	621 + 624 = 624
265	-2	901	623	623	Change sign of 623
266	-2	901	624	624	Change sign of 624

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

267	+2	406	402	625	$(\cos\beta_{12})x(\cos\alpha_{13}) = 625$
268	+2	406	403	626	$(\cos\beta_{12})x(i\ \sin\alpha_{13}) = 626$
269	+2	407	402	627	$(i\ \sin\beta_{12})x(\cos\alpha_{13}) = 627$
270	-2	407	403	628	$-(i\ \sin\beta_{12})x(i\ \sin\alpha_{13}) = 628$
271	+1	626	627	627	626 + 627 = 627
272	+1	625	628	628	625 + 628 = 628
273	-2	901	627	627	Change sign of 627
274	-2	901	628	628	Change sign of 628
275	+1	608	612	629	608 + 612 = 629
276	+1	629	616	629	629 + 616 = 629
277	+1	620	624	630	620 + 624 = 630
278	+1	630	628	630	630 + 628 = 630
279	+1	629	630	630	629 + 630 = 630
280	+1	607	611	631	607 + 611 = 631
281	+1	631	615	631	631 + 615 = 631
282	+1	619	623	632	619 + 623 = 632
283	+1	632	627	632	632 + 627 = 632
284	+1	631	632	632	631 + 632 = 632
285	+2	630	494	633	630 x 494 = 633
286	-2	630	499	634	-630 x 499 = 634
287	+2	632	494	635	632 x 494 = 635
288	+2	632	499	636	632 x 499 = 636
289	+1	634	635	635	634 + 635 = 635
290	+1	633	636	636	633 + 636 = 636

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

291	+3	636	554	637	N_4 Z_4
292	+3	635	554	638	$i\eta_4$
295	+2	701	711	640	Begin finding Resultant σ_{15}
296	+2	704	711	641	β_{15}
296a	+7	300	641	641	Punch β_{15}
297	+0	354	640	642	$\cos\alpha_{15}$
298	-1	642	901	642	$(\cos\alpha_{15} - 1)$
299	+0	353	640	643	$i \sin\alpha_{15}$
300	+0	354	641	644	$\cos\beta_{15}$
301	-1	644	901	644	$(\cos\beta_{15} - 1)$
302	+0	353	641	645	$i \sin\beta_{15}$
303	+2	642	555	646	$(\cos\alpha_{15}-1)N_2$ = 646
304	+2	642	556	647	$(\cos\alpha_{15}-1)i\eta_2$ = 647
305	+2	643	555	648	$(i \sin\alpha_{15})N_2$ = 648
306	-2	643	556	649	$-(i \sin\alpha_{15})i\eta_2$ = 649
307	+1	647	648	650	647 + 648 = 650
308	+1	646	649	651	646 + 649 = 651
309	+2	644	603	652	$(\cos\beta_{15}-1)N_3$ = 652
310	+2	644	604	653	$(\cos\beta_{15}-1)i\eta_3$ = 653
311	+2	645	603	654	$(i \sin\beta_{15})N_3$ = 654
312	-2	645	604	655	$-(i \sin\beta_{15})i\eta_3$ = 655
313	+1	653	654	654	653 + 654 = 654
314	+1	652	655	655	652 + 655 = 655
315	+1	651	655	666	651 + 655 = 666

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

316	+1	650	654	667	650 + 654 = 667
317	-2	901	666	666	Change sign of 666
318	-2	901	667	667	Change sign of 667
319	+2	637	637	668	$(N_4)^2$ = 668
320	+2	638	638	669	$(i\,\eta_4)^2$ = 669
321	+1	668	669	669	668 + 669 = 669
322	+2	666	637	670	666 x 637 = 670
323	-2	666	638	671	-666 x 638 = 671
324	+2	667	637	672	667 x 637 = 672
325	+2	667	638	673	667 x 638 = 673
326	+1	671	672	672	671 + 672 = 672
327	+1	670	673	673	670 + 673 = 673
328	+3	673	669	674	673 ÷ 669 = 674
329	+3	672	669	675	672 ÷ 669 = 675
330	+1	674	901	676	674 + 1.0 = 676
331	+3	675	676	676	675 ÷ 676 = 676
332	+0	355	676	677	Arc tan 676 = 677 Resultant σ_{15}
333	+7	300	677	677	Punch σ_{15}
334	-1	677	710	664	$\sigma_{\text{Resultant}} - \sigma_{\text{Desired}} = \Delta\sigma_{15}$
359	-1	704	713	704	Reduce β_{12} by $\Delta\beta_{12}$
360	-1	705	714	705	Reduce β_{13} by $\Delta\beta_{13}$
361	-1	706	715	706	Reduce β_{14} by $\Delta\beta_{14}$
362	+8	700	664	001	Return to P.P.#1 until $\Delta\sigma_{15} < 0$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Continued)

362a	+2	713	902	675	$\Delta\beta_{12}$ x 2.0 = 675
362b	+2	714	902	676	$\Delta\beta_{13}$ x 2.0 - 676
362c	+2	715	902	677	$\Delta\beta_{14}$ x 2.0 = 677
363	+1	704	675	704	Add 2 x $\Delta\beta_{12}$ back to β_{12}
364	+1	705	676	705	Add 2 x $\Delta\beta_{13}$ back to β_{13}
365	+1	706	677	706	Add 2 x $\Delta\beta_{14}$ back to β_{14}
366	+3	713	905	713	Divide $\Delta\beta_{12}$ by 10
367	+3	714	905	714	Divide $\Delta\beta_{13}$ by 10
368	+3	715	905	715	Divide $\Delta\beta_{14}$ by 10
369	+8	100	004	001	Return to P.P.#1, 4 times then continue
370	+7	300	555	556	Punch Final Z_2
371	+7	300	603	604	Punch Final Z_3
372	+7	300	637	638	Punch Final Z_4
373	0	000	000		Stop

Alternate Card For Card No. 334

334a	-1	710	677	664	$\sigma_{\text{Desired}} - \sigma_{\text{Resultant}} = \Delta\sigma_{15}$

BGPS PROGRAM ORDERS FOR FIVE POSITION CASE (Concluded)

DATA INPUT

374	701	6	+3	000	000	051	α_{12}
	702		+6	000	000	051	α_{13}
	703		+9	000	000	051	α_{14}
	704		+2	500	000	051	β_{12}
	705		+5	000	000	051	β_{13}
	706		+7	500	000	051	β_{14}
375	707	6	+1	550	000	051	$\sigma_{\overline{12}}$
	708		+3	400	000	051	$\sigma_{\overline{13}}$
	709		+5	550	000	051	$\sigma_{\overline{14}}$
	710		+8	000	000	051	$\sigma_{\overline{FR}}$ desired = 80°
	711		+4	000	000	050	4
	712		+1	800	000	052	180 degrees
376	713	3	+1	000	000	050	1 $\Delta\beta_{12}$
	714		+2	000	000	050	2 $\Delta\beta_{13}$
	715		+3	000	000	050	3 $\Delta\beta_{14}$
377	000	0					Start

WORKS CONSULTED

WORKS CONSULTED

1. Translations of Archimedes, "The Works of Archimedes." The Great Books, Encyclopaedia Britannica. 11, 399.

2. Grodzinski, Paul and Ewen McEwen, "Link Mechanisms in Modern Kinematics," Proceedings of the Institute of Mechanical Engineers, vol. 168 (1954), item 37.

3. Hrones, John A., and George L. Nelson. Links and Link-Motion. Technology Press, Massachusetts Institute of Technology and Wiley, New York, 1951.

4. Dunk, Allen C., "Designing 4-Bar Linkages," Machine Design, 27 (February, 1955), 195-199.

5. Schaffer, B. W., and I. Cochin, "Synthesis of 4-Bar Mechanisms When the Position of Two Members is Prescribed," Transactions, American Society of Mechanical Engineers, 76 (1954) 147-156.

6. Freudenstein, Ferdinand, "Approximate Synthesis of 4-Bar Linkages," Transactions, American Society of Mechanical Engineers, 77 (1955) 853-861.

7. Padis, Leonidas A., "Links and Link Motion," Bulletin of the Virginia Polytechnic Institute, vol. 50, no. 10 (August, 1957).

8. Freudenstein, Ferdinand, "4-Bar Function Generators," Transactions, Fifth Conference on Mechanisms, Purdue University (October 13-14, 1958) p. 104.